MA 114 Worksheet #06: Simpson's Rule and Improper Integrals

1. (a) Write down Simpson's rule and illustrate how it works with a sketch.

   (b) How large should n be in the Simpson's rule so that you can approximate

   $$\int_0^1 \sin x \, dx$$

   with an error less than 10^{-7}?

2. Approximate the integral

   $$\int_1^2 \frac{1}{x} \, dx$$

   using Simpson's rule. Choose n so that your error is certain to be less than 10^{-3}. Compute the exact value of the integral and compare to your approximation.

3. Simpson's Rule turns out to exactly integrate polynomials of degree three or less. Show that Simpson's rule gives the *exact* value of $\int_0^h p(x)\, dx$ where $h > 0$ and $p(x) = ax^3 + bx^2 + cx + d$. [Hint: First compute the exact value of the integral by direct integration. Then apply Simpson's rule with $n = 2$ and observe that the approximation and the exact value are the same.]

4. For each of the following, determine if the integral is proper or improper. If it is improper, explain why. Do ***not*** evaluate any of the integrals.

   (a) $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6} \, dx$

   (b) $\displaystyle\int_1^2 \frac{1}{2x - 1} \, dx$

   (c) $\displaystyle\int_1^2 \ln(x - 1) \, dx$

   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1 + x^2} \, dx$

   (e) $\displaystyle\int_0^{\pi/2} \sec x \, dx$

5. For the integrals below, determine if the integral is convergent or divergent. Evaluate the convergent integrals.

   (a) $\displaystyle\int_{-\infty}^0 \frac{1}{2x - 1} \, dx$

   (b) $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2} \, dx$

   (c) $\displaystyle\int_0^2 \frac{x - 3}{2x - 3} \, dx$

   (d) $\displaystyle\int_0^{\infty} \sin\theta \, d\theta$

6. Consider the improper integral

   $$\int_1^{\infty} \frac{1}{x^p} \, dx.$$

Integrate using the generic parameter p to prove the integral converges for $p > 1$ and diverges for $p \leq 1$. You will have to distinguish between the cases when $p = 1$ and $p \neq 1$ when you integrate.

7. Use the Comparison Theorem to determine whether the following integrals are convergent or divergent.
   (a) $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$
   (b) $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^6+x}}\,dx$

8. Explain why the following computation is wrong and determine the correct answer. (Try sketching or graphing the integrand to see where the problem lies.)

$$\begin{aligned}\int_2^{10} \frac{1}{2x-8}dx &= \frac{1}{2}\int_{-4}^{12} \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|x|\Big|_{-4}^{12} \\ &= \frac{1}{2}(\ln 12 - \ln 4)\end{aligned}$$

   where we used the substitution

$$\begin{cases} u(x) = 2x - 8 \\ u(2) = -4 \qquad u(10) = 12 \\ \dfrac{du}{dx} = 2 \end{cases}$$

9. A manufacturer of light bulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the companys bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.
   (a) Make a rough sketch of what you think the graph of F might look like.
   (b) What is the meaning of the derivative $r(t) = F'(t)$?
   (c) What is the value of $\int_0^\infty r(t)\,dt$? Why?