Exercise 1:

Integration by substitution

a) $\int x \cos(x^2)\, dx$

$u = x^2$

$du = dx^2 = (x^2)' dx = 2 \cdot x\, dx$

$\boxed{\frac{du}{2} = x\, dx}$

$\int x \cos(x^2)\, dy = \int \cos u \cdot \frac{du}{2} = \frac{1}{2} \int \cos u\, du = \frac{1}{2} \sin u + c$

$= \frac{1}{2} \sin x^2 + c$

b) $\boxed{\int \underbrace{e^x}_{} \underbrace{\sin x}_{} \, dx}$ $= I$ $\rightarrow$ $\boxed{\text{Integration by parts}}$

$\boxed{u = e^x} \rightarrow du = de^x = e^x \, dx$ $\quad (e^x)'$

$dv = \sin x \, dx \Rightarrow v = \int dv = \int \sin x \, dx =$

$= \boxed{-\cos x = v}$

In general:

$$\int u \, dv = \underbrace{u \cdot v}_{} - \int v \, du$$

$$\int e^x \sin x \, dx = e^x \cdot (-\cos x) - \int (-\cos x) \cdot e^x \, dx = -e^x \cos x + \int \cos x \, e^x \, dx$$

$$\boxed{\int \cos x \, e^{x} \, dx}$$

$$\boxed{u = e^{x}} \rightarrow \boxed{du = e^{x} \, dx}$$

$$\boxed{dv = \cos x \, dx} \rightarrow V = \int dv = \int \cos x \, dx = \boxed{\sin x = V}$$

$$\boxed{\int \cos x \, e^{x} \, dx} = e^{x} \cdot \sin x - \boxed{\int \sin x \, e^{y} \, dx} = I$$

$$I = -e^{x} \cos x + \int \cos x \, e^{y} \, dx = -e^{x} \cos x + \left(e^{x} \sin x - I \right)$$

$$I = -e^{y} \cos x + e^{y} \sin x - I \Rightarrow \boxed{2I = -e^{y} \cos x + e^{x} \sin x}$$

$$I = \int \sin x \, e^{x} \, dx = \frac{-e^{x} \cos x + e^{x} \sin x}{2} + C$$

$$\int e^x \sin x \, dx \qquad \int e^x \cos x \, dx$$

Exercise 1:

c) $\int \frac{\ln(\arctan(x))}{1+x^2}\, dx$

$\int \ln(u)\cdot 1 \cdot du$

In general

$\int u\, dv = u \cdot v - \int v\, du$

$u \to k$, $v \to l$

$u = \arctan(x)$

$du = d(\arctan(x)) = (\arctan(x))' \cdot dx = \frac{dx}{1+x^2} = du$

$k = \ln u \to dk = d(\ln u) = (\ln u)'\, du = \frac{1}{u}\, du$

$dl = 1 \cdot du$

$\to l = \int dl = \int du = \boxed{u = l}$

$\int \ln u\, du = \ln(u) \cdot u - \int u \cdot \frac{du}{u} = \ln(u) \cdot u - \int du$

$$\int \ln u \, du = \ln u \cdot u - \int du = \ln u \cdot u - u + C$$

$$\int \frac{\ln(\arctan(x))}{1+x^2} \, dx = \ln(\arctan(x)) \cdot \arctan(x) - \arctan(x) + C$$

Exercise 2 :

a) $\int x^2 \sin(x)\,dx$ $\nearrow$ Integration by parts

$u = x^2 \rightarrow \boxed{du = 2x\,dx}$

$dv = \sin x\,dx \rightarrow v = \int dv = \int \sin x\,dx =$

$= -\cos x$

$\int x^2 \sin(x)\,dx = x^2 \cdot (-\cos x) - \int (-\cos x) \cdot 2x\,dx =$

$= -x^2 \cos x + \boxed{\int 2x \cos x\,dx}$

$u = 2x \rightarrow du = 2\,dx$

$dv = \cos x\,dx \rightarrow v = \int dv = \int \cos x\,dx = \sin x$

$$\int 2x\cos(x)\,dx = 2x\cdot\sin(x) - \int \sin(x)\cdot 2\,dx =$$

$$= 2x\cdot\sin(x) - 2\int \sin(x)\,dx = 2\cdot x\sin(x) - 2\cdot(-\cos x) + C$$

$$\int x^2\sin(x)\,dx = -x^2\cdot\cos x + 2x\sin(x) + 2\cos(x) + C$$

Exercise 2 :

$$g) \int x \ln(1+x)\, dx$$

$u = 1 + x \Rightarrow x = u - 1$

$du = (1+x)'\, dx = dx$

$(1+x)' = 1' + x' = 0 + 1 = 1$

$$\int x \ln(1+x)\, dx = \int (u-1) \ln u \cdot du$$

$k = \ln u \longrightarrow dk = \frac{1}{u}\, du$

$dl = (u-1)\, du \rightarrow l = \int dl = \int (u-1)\, du = \int u\, du - \int 1 \cdot du = \frac{u^2}{2} - u = l$

$$\int (u-1)\ln(u)\,du = \ln(u)\cdot\left(\frac{u^2}{2}-u\right) - \int\left(\frac{u^2}{2}-u\right)\frac{1}{u}\,du$$

$$= \ln(u)\cdot\left(\frac{u^2}{2}-u\right) - \int\left(\frac{u^{\cancel{2}}}{2\cdot\cancel{u}} - \frac{\cancel{u}}{\cancel{u}}\right)du =$$

$$= \ln(u)\cdot\left(\frac{u^2}{2}-u\right) - \int\left(\frac{u}{2}-1\right)du = \ln(u)\cdot\left(\frac{u^2}{2}-u\right) -$$

$$-\int\frac{u}{2}\,du + \int du = \ln(u)\cdot\left(\frac{u^2}{2}-u\right) - \frac{1}{2}\cdot\frac{u^2}{2} + u + C$$

$$= \ln(x+1)\cdot\left(\frac{(x+1)^2}{2}-(x+1)\right) - \frac{1}{4}(x+1)^2 + (x+1) + C$$

$$\int x \ln(1+x)\,dx = \ln(x+1)\cdot\left(\frac{(x+1)^2}{2} - (x+1)\right)$$

$$- \frac{1}{4}(x+1)^2 + (x+1) + C$$

Exercise 4 :

$f(1) = 2$, $f(4) = 7$ $f'(1) = 5$

$f'(4) = 3$

$$\int_1^4 x f''(x)\,dx$$

$u = x \rightarrow \boxed{du = dx}$

$dv = f''(x)\,dx \rightarrow v = \int dv = \int f''(x)\,dx$

$= \boxed{f'(x) = v}$

$$\int_1^4 x f''(x)\,dx = \left(x \cdot f'(x)\right)\Big|_1^4 - \int_1^4 f'(x)\,dx =$$

$$= \left(4 \cdot f'(4) - 1 \cdot f'(1)\right) - f(x)\Big|_1^4 = (4 \cdot 3 - 1 \cdot 5) - (f(4) - f(1))$$

$$= (12 - 5) - (7 - 2) =$$

$$= 7 - 5 = \boxed{2}$$

$$\int_0^0 x\,dx = \frac{x^2}{2} + C$$

$$\int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = \left(\frac{2^2}{2} - \frac{1^2}{2}\right)$$

Exercise 3:

$$\int_0^3 x \sin(3-x)\,dx$$

$$\boxed{l = 3 - x} \Rightarrow l - 3 = -x$$

$$x = 3 - l$$

$$dl = (3-x)'\,dx = -dx = dl$$

multiply by (-1)

$\to x = 0 \to l = 3 - 0 = 3$

$\to x = 3 \to l = 3 - 3 = 0$

$$\int_3^0 (3-l)\sin l\,(-dl) = \int_0^3 (3-l)\sin(l)\,dl$$

Integration by parts

$$u = 3 - \ell \rightarrow du = -d\ell$$

$$dv = \sin \ell \rightarrow v = \int dv = \int \sin \ell \, d\ell = -\cos \ell$$

$$\int_0^3 (3-\ell) \sin \ell \, d\ell = (3-\ell)\cdot(-\cos \ell)\Big|_0^3 - \int_0^3 -\cos \ell \cdot -d\ell =$$

$$= \Big(\underbrace{\overset{=0}{(3-3)} \cdot (-\cos(3))}_{F(3)} - \overbrace{(3-0)\cdot(-\cos 0)}^{F(1)} \Big) - \int_0^3 \cos \ell \, d\ell =$$

$$= 3 \cdot 1 - \sin \ell \Big|_0^3 = 3 - \Big(\sin(3) - \overset{=0}{\sin(0)} \Big) = 3 - \sin 3 + \overset{=0}{\sin 0}$$

Exercise 1 :

a) $\int \sin(x) \sec^2(x)\, dx = \int \sin(x) \frac{1}{\cos^2(x)}\, dx$

$$\boxed{u = \cos x}$$

$$du = -\sin x\, dx$$

$$\Downarrow$$

$$\boxed{-du = \sin(x)\, dx}$$

$$= \int \frac{1}{u^2} \cdot -du = -\int \frac{1}{u^2}\, du = -\int u^{-2}\, du = -\frac{u^{-2+1}}{-2+1} + c$$

$$= -\frac{u^{-1}}{-1} + c = +(\cos(x))^{-1} + c$$

---

note: $\int x^a\, dx = \frac{x^{a+1}}{a+1} + c$ $\qquad \int \frac{1}{x^a}\, dx = \int x^{-a}\, dx = \frac{x^{-a+1}}{-a+1} + c$

$a = 2, 3, 4, \ldots$

Exercise: Compute the following integrals:

a) $\boxed{\int \sin(x)\sec^2(x)\,dx} = \int \sin(x)\,\frac{1}{\cos^2(x)}\,dx$

$\boxed{u = \cos x}$

$du = -\sin x\,dx$

$\Downarrow$

$\boxed{-du = \sin(x)\,dx}$

$= \int \frac{1}{u^2}\cdot -du = -\int \frac{1}{u^2}\,du = -\int u^{-2}\,du = -\frac{u^{-2+1}}{-2+1} + c$

$= -\frac{u^{-1}}{-1} + c = +(\cos(x))^{-1} + c = \frac{1}{\cos(x)} + c$

note: $\int x^a\,dx = \frac{x^{a+1}}{a+1} + c$ $\qquad \int \frac{1}{x^a}\,dx = \int x^{-a}\,dx = \frac{x^{-a+1}}{-a+1} + c$

$a = 2, 3, 4, \ldots$

Exercise 1: Compute the integrals:

g) $\int 4 \sin^2(x) \cos^2(x)\,dx$

$$\boxed{2\sin(x)\cdot\cos(x) = \sin(2x)}$$

$$\boxed{4\sin^2(x)\cdot\cos^2(x)} = (2\sin(x)\cdot\cos(x))(2\sin(x)\cos(x)) = \sin^2(2x)$$

$$= \int \sin^2(2x)\,dx = I$$

$$\boxed{u = 2x} \to \boxed{du = d(2x) = (2x)'\cdot dx = 2\,dx} \Rightarrow du = 2\,dx$$

$$\boxed{dx = \frac{1}{2}\,du}$$

$$I = \int \sin^2(u)\,\frac{1}{2}\,du = \frac{1}{2}\int \sin^2(u)\,du$$

$$I = \frac{1}{2}\int \sin^2(u)\,du$$

$$\boxed{\sin^2(x) + \cos^2(x) = 1} \to \boxed{\cos^2 x = 1 - \sin^2 x}$$

$$\boxed{\cos(2x) = \cos^2(x) - \sin^2(x)}$$

$$= 1 - \sin^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$

$$\cos(2x) = 1 - 2\sin^2(x) \to$$

$$\cos(2x) - 1 = -2\sin^2(x) \Rightarrow$$

$$\sin^2(x) = \frac{\cos(2x) - 1}{-2} = \frac{1 - \cos(2x)}{2}$$

$$I = \frac{1}{2}\int \frac{1 - \cos(2u)}{2}\,du = \frac{1}{4}\int\left(1 - \cos(2u)\right)du =$$

Exercise 1:

a) $\int \sin(x) \sec^2(x)\,dx = \int \sin(x) \frac{1}{\cos^2(x)}\,dx$

$\boxed{u = \cos x}$

$du = -\sin x\,dx$

$\Downarrow$

$\boxed{-du = \sin(x)\,dx}$

$= \int \frac{1}{u^2} \cdot -du = -\int \frac{1}{u^2}\,du = -\int u^{-2}\,du = -\frac{u^{-2+1}}{-2+1} + c$

$= -\frac{u^{-1}}{-1} + c = + (\cos(x))^{-1} + c$

note: $\int x^a\,dx = \frac{x^{a+1}}{a+1} + c$ $\qquad \int \frac{1}{x^a}\,dx = \int x^{-a}\,dx = \frac{x^{-a+1}}{-a+1} + c$

$a = 2, 3, 4, \ldots$

Exercise 1 Compute the following integrals:

a) $\boxed{\int \sin(x)\sec^2(x)\,dx} = \int \sin(x)\,\frac{1}{\cos^2(x)}\,dx$

$$\boxed{u = \cos x}$$

$$du = -\sin x\,dx$$

$$\Downarrow$$

$$\boxed{-du = \sin(x)\,dx}$$

$$= \int \frac{1}{u^2}\cdot -du = -\int \frac{1}{u^2}\,du = -\int u^{-2}\,du = -\frac{u^{-2+1}}{-2+1} + c$$

$$= -\frac{u^{-1}}{-1} + c = +(\cos(x))^{-1} + c = \frac{1}{\cos(x)} + c$$

note: $\int x^a\,dx = \frac{x^{a+1}}{a+1} + c$

$$\int \frac{1}{x^a}\,dx = \int x^{-a}\,dx = \frac{x^{-a+1}}{-a+1} + c$$

$a = 2, 3, 4, \ldots$

Exercise 1 : Compute the integrals :

$$\boxed{2\sin(x)\cdot\cos(x) = \sin(2x)}$$

g) $\int 4\sin^2(x)\cos^2(x)\,dx$

$$\boxed{4\sin^2(x)\cdot\cos^2(x)} = (2\sin(x)\cdot\cos(x))(2\sin(x)\cos(x)) = \sin^2(2x)$$

$= \int \sin^2(2x)\,dx = I$

$\boxed{u = 2x} \rightarrow \boxed{du = d(2x) = (2x)'\cdot dx = 2\,dx} \Rightarrow du = 2\,dx$

$$\boxed{dx = \frac{1}{2}\,du}$$

$I = \int \sin^2(u)\,\frac{1}{2}\,du = \frac{1}{2}\int \sin^2(u)\,du$

$$I = \frac{1}{2}\int \sin^2(u)\,du$$

$$\boxed{\sin^2(x) + \cos^2(x) = 1} \rightarrow \boxed{\cos^2 x = 1 - \sin^2 x}$$

$$\boxed{\cos(2x) = \cos^2(x) - \sin^2(x)}$$

$$= 1 - \sin^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$

$$\cos(2x) = 1 - 2\sin^2(x) \rightarrow$$

$$\cos(2x) - 1 = -2\sin^2(x) \Rightarrow$$

$$\sin^2(x) = \frac{\cos(2x) - 1}{-2} = \frac{1 - \cos(2x)}{2}$$

$$I = \frac{1}{2}\int \frac{1 - \cos(2u)}{2}\,du = \frac{1}{4}\int \left(1 - \cos(2u)\right)du =$$

Note

$$\int \sin(2x)\,dx$$

$u = 2x$

$du = 2\,dx \Rightarrow$

$\frac{du}{2} = dx$

$$\int \sin(u)\,\frac{1}{2}\,du = -\frac{1}{2}\cos(u) + C$$

$u = \sin x$

$$\boxed{\int \underbrace{\sin^2(x)}\,d\underbrace{(\sin(x))}} = \int u^2\,du = \frac{u^{2+1}}{2+1} + C = \frac{(\sin(x))^3}{3} + C$$

$$I = \frac{1}{4}\int \left(1 - \cos(2u)\right) du = \frac{1}{4}\int du - \frac{1}{4}\int \cos(2u)\, du$$

$$\boxed{2u = t} \Rightarrow dt = d(2u) = (2u)'\, du = 2\, du$$

$$\Rightarrow u = \frac{t}{2} \qquad \boxed{du = \frac{dt}{2}}$$

$$I = \frac{1}{4}\cdot\frac{t}{2} - \frac{1}{4}\int \cos(t)\cdot\frac{dt}{2} = \frac{1}{4}\,\frac{t}{2} - \frac{1}{4}\cdot\frac{1}{2}\int \cos(t)\, dt$$

$$= \frac{1}{4}\,\frac{t}{2} - \frac{1}{8}\sin t + C = \frac{1}{8}\cdot 2u - \frac{1}{8}\sin(2u) + C$$

$$= \boxed{\frac{1}{8}\cdot 2\cdot 2x - \frac{1}{8}\sin(4x) + C}$$

a) $\int x \cos(x^2)\, dx$

$du = dx^2 = (x^2)' \cdot dx$

$= 2x\, dx$

$u = x^2$

$= \int \frac{2}{2} x \cos(x^2)\, dx =$

$= \frac{1}{2} \int \cos(x^2)\, 2x\, dx = \frac{1}{2} \int \cos(x^2)\, dx^2 = \frac{1}{2} \int \cos u\, du =$

$= \frac{1}{2} \sin u + C$

b) $\int e^x \sin x\, dx$ → integration by part

dv

$\boxed{u = e^x}$ → $du = e^x dx$

$\boxed{dv = \sin x\, dx}$ → $v = \int dv = \int \sin x\, dx = -\cos x$

In general

$\int u\, dv = u \cdot v - \int v\, du$

$$\underbrace{\int e^x \sin x\,dx}_{I} = e^x \cdot (-\cos x) - \int (-\cos x) \cdot e^x\,dx =$$

$$= -e^x \cos x + \underbrace{\int \cos x\, e^x\,dx}_{J}$$

$u = e^x \to du = e^x\,dx$

$dv = \cos x\,dx \to v = \int \cos x\,dx = \sin x$

$\int u\,dv = u \cdot v - \int v\,du$

$J = e^x \cdot \sin x - \int \sin x\, e^x\,dx$

$I = -e^x \cos x + e^x \sin x - I \Rightarrow 2I = -e^x \cos x + e^x \sin x$

$$I = \frac{-e^x \cos x + e^x \sin x}{2} + C$$

a) $\int x \cos(x^2)\, dx$

$du = dx^2 = (x^2)' \cdot dx$

$= 2x\, dx$

$u = x^2$

$= \int \frac{2}{2} x \cos(x^2)\, dx =$

$= \frac{1}{2} \int \cos(x^2)\, 2x\, dx = \frac{1}{2} \int \cos(x^2)\, dx^2 = \frac{1}{2} \int \cos u\, du =$

$= \frac{1}{2} \sin u + C$

b) $\int e^x \underbrace{\sin x\, dx}_{dv}$ → integration by part

$\boxed{u = e^x}$ → $du = e^x\, dx$

$\boxed{dv = \sin x\, dx}$ → $v = \int dv = \int \sin x\, dx = -\cos x$

In general

$\int u\, dv = u \cdot v - \int v\, du$

$$\underbrace{\boxed{\int e^x \sin x\,dx}}_{I} = e^x \cdot (-\cos x) - \int (-\cos x) \cdot e^x\,dx =$$

$$= -e^x \cos x + \underbrace{\boxed{\int \cos x\, e^x\,dx}}_{J}$$

$$\int u\,dv = u \cdot v - \int v\,du$$

$u = e^x \longrightarrow du = e^x\,dx$

$dv = \cos x\,dx \rightarrow v = \int \cos x\,dx = \sin x$

$$J = e^x \cdot \sin x - \int \sin x\, e^x\,dx$$

$$I = -e^x \cos x + e^x \sin x - I \Rightarrow 2I = -e^x \cos x + e^x \sin x$$

$$I = \frac{-e^x \cos x + e^x \sin x}{2} + C$$

a) $\int x \cos(x^2)\,dx$

$$\boxed{u = x^2}$$

$dx^2 = (x^2)' \cdot dx$ $du = 2x\,dx$

$= 2x\,dx \Rightarrow$ $\boxed{\frac{du}{2} = x\,dx}$

$\int \cos x\,dx = \sin x$

$$\int \cos x^2\, x\,dx = \int \cos u \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C$$

$$= \frac{1}{2}\sin x^2 + C$$

a) $\int x \cos(x^2)\,dx$

$du = dx^2 = (x^2)' \cdot dx$

$= 2x\,dx$

$u = x^2$

$= \int \frac{2}{2} x \cos(x^2)\,dx =$

$= \frac{1}{2} \int \cos(x^2)\, 2x\,dx = \frac{1}{2} \int \cos(x^2)\,dx^2 = \frac{1}{2} \int \cos u\,du =$

$= \frac{1}{2} \sin u + C$

b) $\int e^x \underbrace{\sin x\,dx}_{dv}$ → integration by part

$\boxed{u = e^x}$ → $du = e^x\,dx$

$\boxed{dv = \sin x\,dx}$ → $v = \int dv = \int \sin x\,dx = -\cos x$

In general

$\int u\,dv = u \cdot v - \int v\,du$

$$\underbrace{\int e^x \sin x\,dx}_{I} = e^x \cdot (-\cos x) - \int (-\cos x) \cdot e^x\,dx =$$

$$= -e^x \cos x + \underbrace{\int \cos x\, e^x\,dx}_{J}$$

$$\int u\,dv = u \cdot v - \int v\,du$$

$u = e^x \rightarrow du = e^x\,dx$

$dv = \cos x\,dx \rightarrow v = \int \cos x\,dx = \sin x$

$J = e^x \cdot \sin x - \int \sin x\, e^x\,dx$

$I = -e^x \cos x + e^x \sin x - I$

$\Rightarrow 2I = -e^x \cos x + e^x \sin x$

$$I = \frac{-e^x \cos x + e^x \sin x}{2} + C$$

a) $\int x \cos(x^2)\, dx$

$\boxed{u = x^2}$

$dx^2 = (x^2)' \cdot dx$ $\quad du = 2x\, dx$

$= 2x\, dx \Rightarrow$ $\boxed{\frac{du}{2} = x\, dx}$

$\int \cos x\, dx = \sin x$

$$\int \cos x^2\, x\, dx = \int \cos u \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int \cos u\, du = \frac{1}{2} \sin u + C$$

$$= \frac{1}{2} \sin x^2 + C$$

$\int 4x \sin x \, dx$

$\int x \cos x^2 \, dx$

$u = 4x \rightsquigarrow du = 4\,dx$

$dv = \sin x \, dx \longrightarrow v = \int dv = \int \sin x \, dx$

Exercise 2:

$$\int u\,dv = u \cdot v - \int v\,du$$

a) $\int x^2 \sin x\, dx$

x

$(x^2)'$

$u = x^2 \longrightarrow du = 2x\, dx$

$dv = \sin x\, dx \longrightarrow v = \int dv = \int \sin x\, dx = -\cos x$

a) $\int x \cos(x^2)\, dx$

$du = dx^2 = (x^2)' \cdot dx$

$= 2x\, dx$

$u = x^2$

$= \int \frac{2}{2} x \cos(x^2)\, dx =$

$= \frac{1}{2} \int \cos(x^2)\, 2x\, dx = \frac{1}{2} \int \cos(x^2)\, dx^2 = \frac{1}{2} \int \cos u\, du =$

$= \frac{1}{2} \sin u + C$

b) $\int e^x \underbrace{\sin x\, dx}_{dv}$ → integration by part

$\boxed{u = e^x}$ → $du = e^x dx$

$\boxed{dv = \sin x\, dx}$ → $v = \int dv = \int \sin x\, dx = -\cos x$

In general

$\int u\, dv = u \cdot v - \int v\, du$

$$\underbrace{\int e^x \sin x\,dx}_{I} = e^x\cdot(-\cos x) - \int(-\cos x)\cdot e^x\,dx =$$

$$= -e^x\cos x + \underbrace{\int \cos x\, e^x\,dx}_{J}$$

$$\int u\,dv = u\cdot v - \int v\,du$$

$u = e^x \rightarrow du = e^x\,dx$

$dv = \cos x\,dx \rightarrow v = \int \cos x\,dx = \sin x$

$$J = e^x\cdot\sin x - \int \sin x\, e^x\,dx$$

$$I = -e^x\cos x + e^x\sin x - I \Rightarrow 2I = -e^x\cos x + e^x\sin x$$

$$I = \frac{-e^x\cos x + e^x\sin x}{2} + C$$

a) $\int x \cos(x^2)\, dx$

$\boxed{u = x^2}$

$dx^2 = (x^2)' \cdot dx$ $\quad du = 2x\,dx$

$= 2x\,dx \Rightarrow \boxed{\frac{du}{2} = x\,dx}$

$\int \cos x\, dx = \sin x$

$\int \cos x^2\, x\, dx = \int \cos u \cdot \frac{du}{2}$

$= \frac{1}{2} \int \cos u\, du = \frac{1}{2} \sin u + C$

$= \frac{1}{2} \sin x^2 + C$

$$\int 4x \sin x \, dx$$

$$\int x \cos x^2 \, dx$$

$u = 4x \rightsquigarrow du = 4\,dx$

$dv = \sin x \, dx \longrightarrow v = \int dv = \int \sin x \, dx$