Worksheet 29:

Exercise 1:

(a) $x^2 = 4y - 2y^2 \iff x^2 + 2y^2 - 4y = 0 \iff$

$x^2 + 2\cdot(y^2 - 2y + 1) - 2 = 0 \iff$

$x^2 + 2\cdot(y-1)^2 = 2 \iff$ $\boxed{\frac{x^2}{2} + \frac{(y-1)^2}{1} = 1}$

$\downarrow$ ellipse

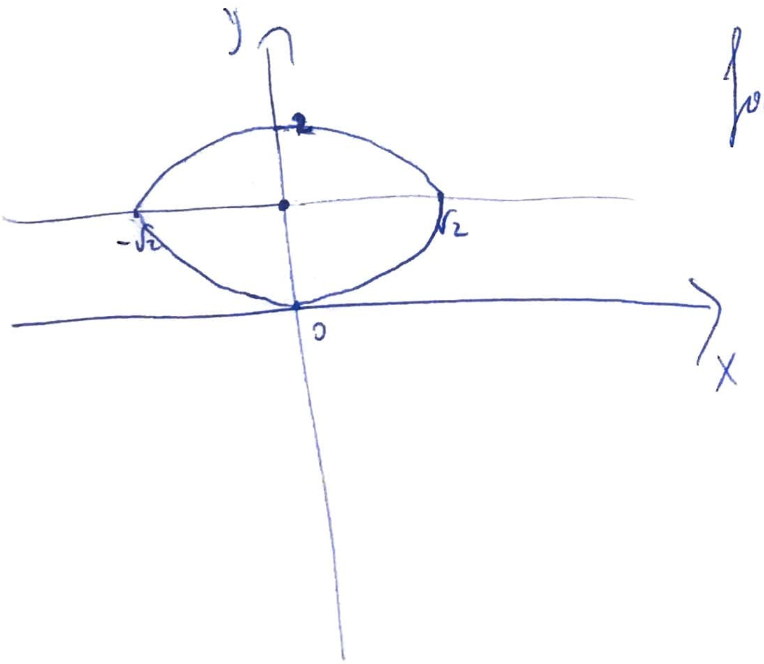
$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y-1}{1}\right)^2 = 1$

Vertices at $(\pm a, 0) \iff$
$(\pm\sqrt{2}, 0)$

foci: $c = \sqrt{a^2 - b^2} = \sqrt{2-1} = 1 \Rightarrow$

foci $(\pm c, 0) \iff (\pm 1, 0)$



(b) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$(x^2 + 2x + 1) - 1 + 3(y^2 - 4y + 4) - 12 + 10 = 0.$

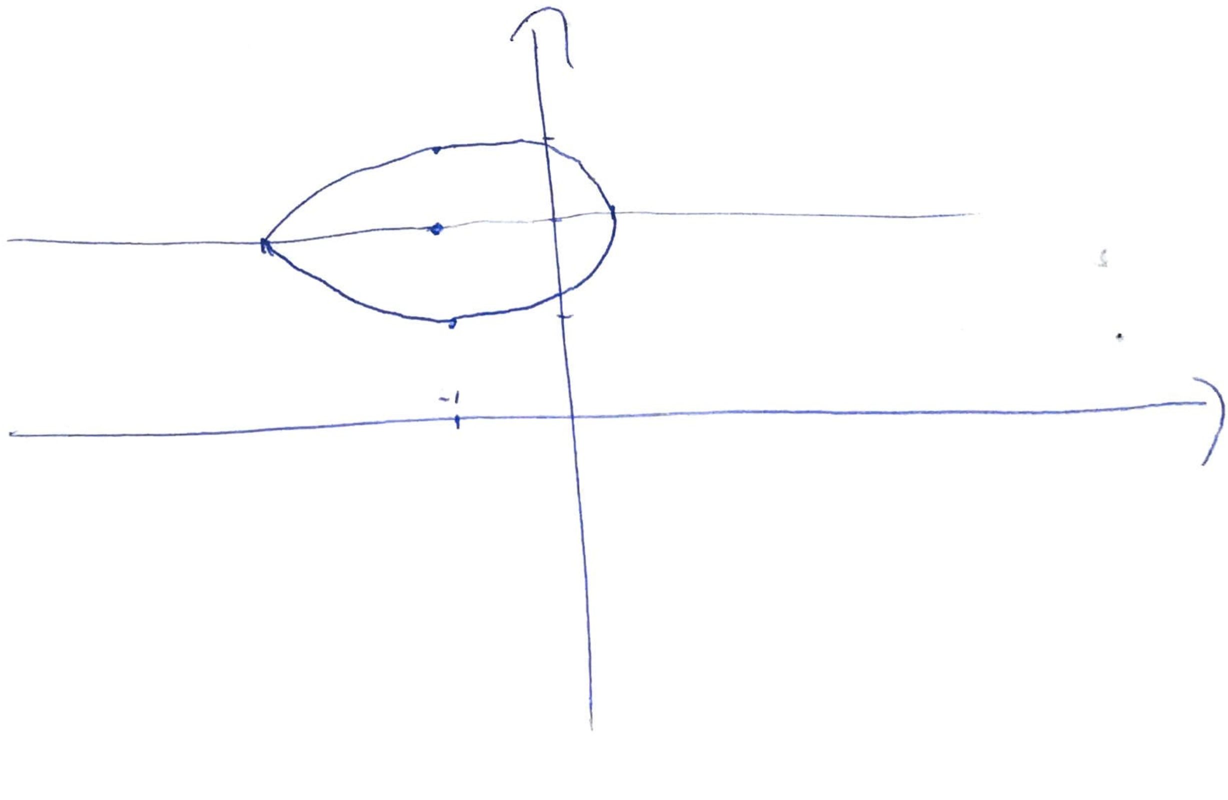
$(x+1)^2 + 3(y-2)^2 = 1 + 12 - 10 = 3 \iff$

$$\boxed{\left(\frac{x+1}{\sqrt{3}}\right)^2 + \left(\frac{y-2}{1}\right)^2 = 1}$$

$a = \sqrt{3}$, $b = 1$ $\Rightarrow$ $c = \sqrt{3-1} = \sqrt{2}$ $\Rightarrow$

foci $(\pm\sqrt{2}, 0)$

vertices $(\pm\sqrt{3}, 0)$.



Exercise 2 :

$$\frac{dy}{dx} = \frac{x \sin(x)}{y} \qquad y(0) = -1.$$

$$\Rightarrow y\, dy = x \sin(x)\, dx \quad \Leftrightarrow \quad \int y\, dy = \int x \sin(x)\, dx$$

$$\frac{y^2}{2} = \sin(x) - x\cos(x) + C. \quad \Leftrightarrow \quad y^2 = 2\sin(x) - 2x\cos(x) + C$$

$$y = \sqrt{2\sin(x) - 2x\cos(x) + C} \quad \Rightarrow$$

$$y(0) = -1 \quad \Rightarrow \quad \sqrt{2\sin(0) - 2\cdot 0 \cdot \cos(0) + C} = -1 \quad \Leftrightarrow$$

$$C = (-1)^2 \quad \Rightarrow \quad \boxed{C = 1} \quad \Rightarrow$$

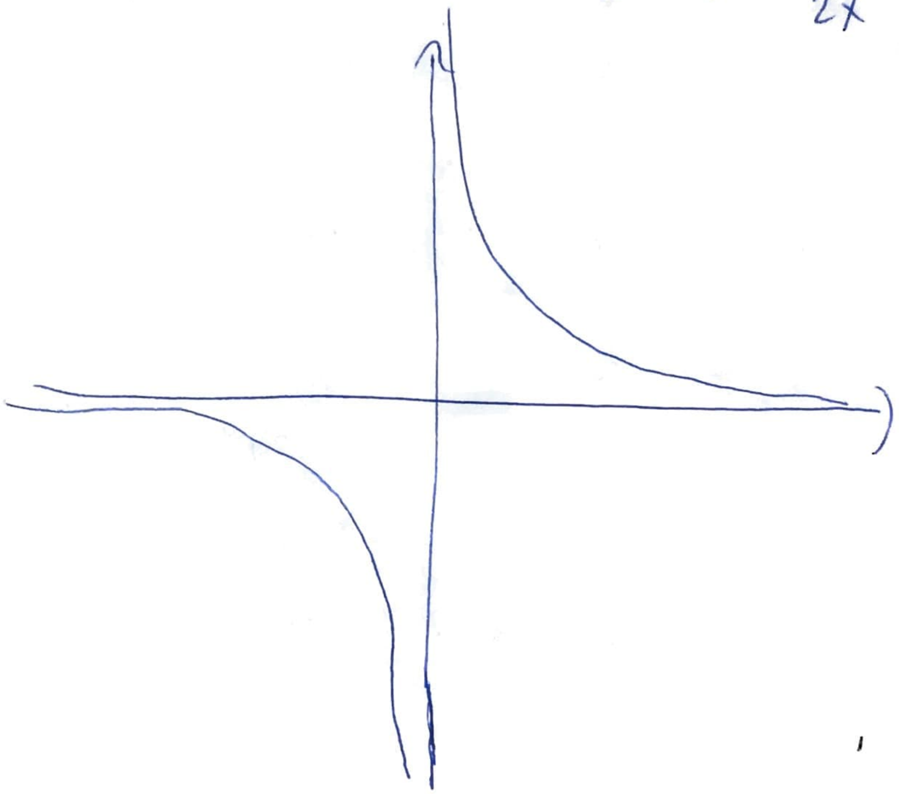
Exercise 3 :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

By using the fact that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ we have that.

$$r^2 \sin 2\theta = 1 \iff 2 \cdot x y = 1 \iff y = \frac{1}{2x}.$$

Using Desmos,



Exercise 5:

Find the slope of the tangent line to $r = 2\cos(\theta)$ at $\theta = \frac{\pi}{3}$.

$$x = r\cos(\theta) = 2\cos(\theta)\cdot\cos(\theta) = 2\cos^2(\theta)$$

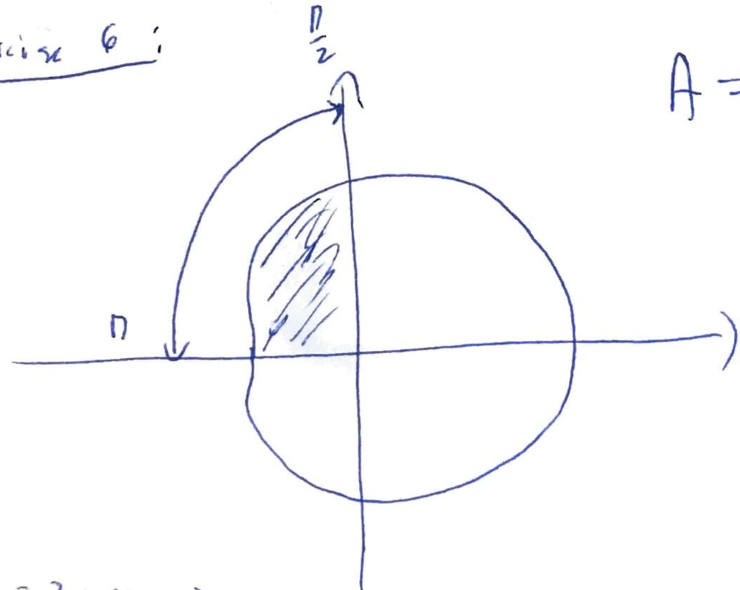
$$y = r\sin(\theta) = 2\cos(\theta)\cdot\sin(\theta) = \sin(2\theta).$$

$$x' = -2\cdot 2\cdot\cos(\theta)\cdot\sin(\theta) = -2\sin(2\theta).$$

$$y' = 2\cdot\cos(2\theta)$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{y'(\theta)}{x'(\theta)}\right|_{\theta=\frac{\pi}{3}} = \frac{-2\sin\left(\frac{2\pi}{3}\right)}{2\cos\left(\frac{2\pi}{3}\right)} = -1\cdot -\sqrt{3} = \sqrt{3}.$$

Exercise 6:



$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta =$$

$$A = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\cdot\left(2+\cos(\theta)\right)^2 d\theta.$$

$r = 2 + \cos(\theta).$

Exercise 7 :

$r = \theta^2$, $0 \le \theta \le 2\pi$ $\Rightarrow$ length of the polar curves,

General formula for Arc length is $\int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$.

we have $r = f(\theta) = \frac{x}{\cos(\theta)} = \frac{y}{\sin(\theta)}$. $\Leftrightarrow$

$$S = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta.$$

$u = \theta^2 + 4$

$du = 2\theta\, d\theta$

$\frac{du}{2} = \theta\, d\theta$

$$= \int_4^{4\pi^2+4} \frac{1}{2}\sqrt{u}\, du = \boxed{\frac{1}{3}\left(4\pi^2+4\right)^{3/2} - \frac{1}{3}\, 4^{\frac{3}{2}}}.$$

Exercise 8 :

$\Delta x = 0,1$, to estimate $y(0,5)$.

$y' = y + xy \quad y(0) = 1.$

We can see that since

$x_0 = 0$

$x_1 = 0 + 0,1 = 0,1$

$x_2 = 0,1 + 0,1 = 0,2$

$x_3 = 0,2 + 0,1 = 0,3$

$y_0 = 1$

$x_4 = 0,3 + 0,1 = 0,4$

$x_5 = 0,4 + 0,1 = 0,5 \Rightarrow y(0,5) = y_5$

$f(x_n, y_n) = y_n + x_n y_n \Leftrightarrow$

$y_{n+1} = y_n + \Delta x \, f(x_n, y_n) \Leftrightarrow$

$y_1 = y_0 + \Delta x \cdot f(x_0, y_0) = y_0 + \Delta x \cdot (y_0 + x_0 y_0) = 1 + 0,1 \cdot (1 + 0 \cdot 1) = 1,1$

$y_2 = y_1 + \Delta x \cdot (y_1 + x_1 \cdot y_1) = 1,1 + 0,1 \cdot (1,1 + 0,11) = 1,221$

$y_3 = y_2 + \Delta x \cdot (y_2 + x_2 y_2) = 1,1 + 0,1 \cdot (1,221 + 0,2442) = 1,24652$

$y_4 = y_3 + \Delta x \cdot (y_3 + x_3 \cdot y_3) = 1,24652 + 0,1 (1,24652 + 0,373956)$

$= 1,4085676.$

$y_5 = y_4 + \Delta x \cdot (y_4 + x_4 y_4) = 1,60576706$

Exercise 8:

$\Delta x = 0,2$ to estimate $y(1)$, $y' = x^2 y - \frac{1}{2} y^2$, $y(0) = 1$.

$\Rightarrow x_0 = 0 \Rightarrow x_1 = x_0 + \Delta x = 0 + 0,2 = 0,2$

$y_0 = 1$

$x_2 = 0,2 + 0,2 = 0,4$

$x_3 = 0,4 + 0,2 = 0,6$

$x_4 = 0,6 + 0,2 = 0,8$

$x_5 = 0,8 + 0,2 = \boxed{1}$ $\Rightarrow$ we need $y(1) = y_5$.

$f(x_n, y_n) = x_n^2 y_n - \frac{1}{2} y_n^2$ and then use

$y_{n+1} = y_n + \Delta x \cdot f(x_n, y_n)$

$$\boxed{y_{n+1} = y_n + \Delta x \cdot \left(x_n^2 y_n - \frac{1}{2} y_n^2 \right)}$$

Exercise 10 :

(a) $\dfrac{dy}{dx} = 3x^2y^2 \iff \dfrac{dy}{y^2} = 3x^2\,dx \iff$

$\displaystyle\int \frac{dy}{y^2} = \int 3x^2\,dx \iff -\frac{1}{y} = 3\frac{x^3}{3} + c = x^3 + c \iff$

$\boxed{y = -\dfrac{1}{x^3+c}}$ or $\boxed{-\dfrac{1}{y} = x^3 + c}$

(b) $xyy' = x^2+1 \iff xy\dfrac{dy}{dx} = x^2+1 \iff$

$y\,dy = \dfrac{x^2+1}{x}\,dx \iff \displaystyle\int y\,dy = \int \frac{x^2+1}{x}\,dx = \int\left(x + \frac{1}{x}\right)dx \iff$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + \ln|x| + c \Rightarrow \boxed{y^2 = x^2 + 2\ln|x| + c}$

(c) $\dfrac{dy}{dx} + e^{x+y} = 0 \iff \dfrac{dy}{dx} = -e^x e^y \iff -\dfrac{dy}{e^y} = e^x\,dx \iff$

$\displaystyle\int e^{-y}\,d(-y) = \int e^x\,dx \iff e^{-y} = e^x + c \iff e^{-y} = e^{cx} \iff$

$-y = cx \iff \boxed{y = \dfrac{1}{c\cdot x}}$

Exercise 11 :

(1) $y' = 2x\sqrt{1-y^2} \iff \dfrac{dy}{dx} = 2x\sqrt{1-y^2} \iff$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx$$

let $y = \sin(t) \iff$

$dy = \cos(t)\,dt \iff$

$\boxed{t = \arcsin(y)}$

$$\int \frac{\cos(t)\,dt}{\sqrt{1-\sin^2(t)}} = 2\frac{x^2}{2} + c = x^2 + c \iff$$

$$\int \frac{\cos(t)\,dt}{\sqrt{\cos^2(t)}} = x^2 + c \iff \int dt = x^2 + c \iff$$

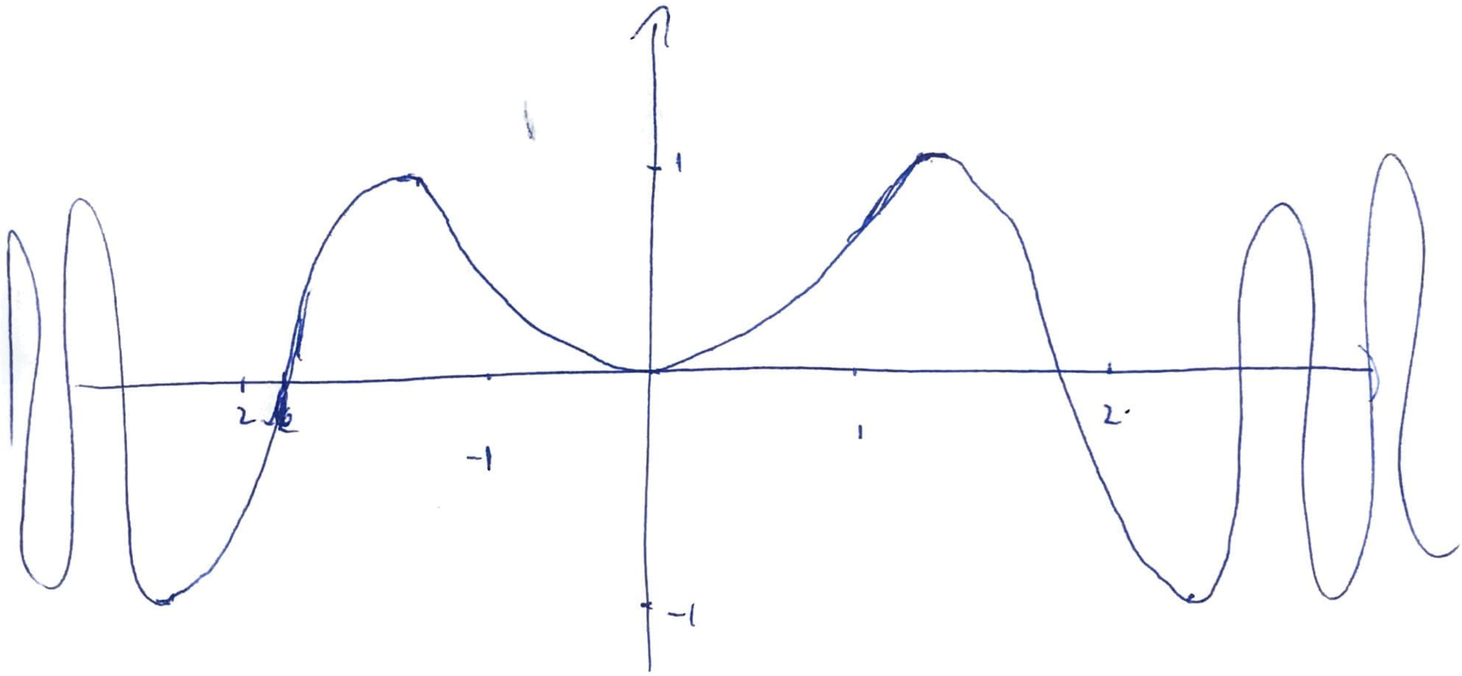
$$t = x^2 + c \iff \arcsin(y) = x^2 + c$$

$$\iff \boxed{y = \sin(x^2 + c)}$$

(b) $y' = 2x\sqrt{1-y^2}$ $y(0) = 0$ $\Leftrightarrow$

from before $y = \sin(x^2 + c)$ $\Leftrightarrow$ $y(0) = 0$ $\Leftrightarrow$.

$0 = \sin(c)$ $\Leftrightarrow$ $c = 0$ $\Leftrightarrow$ $y = \sin(x^2)$.



(c) $y' = 2x\sqrt{1-y^2}$, $y(0) = 2$?.

from before we have $y = \sin(x^2 + c)$ $\Leftrightarrow$

$y(0) = 2$ $\Leftrightarrow$ $2 = \sin(c)$ this is not

possible since $\boxed{-1 \leq \sin(x) \leq 1}$

Exercise 4 :

$y' = y(1-y) \iff$ direction field :

$y \backslash x$	-2	-1	0	1	2
-2	-6	-6	-6	-6	-6
-1	-2	-2	-2	-2	-2
0	1	1	1	1	1
1	0	0	0	0	0
2	-1	-1	-1	-1	-1

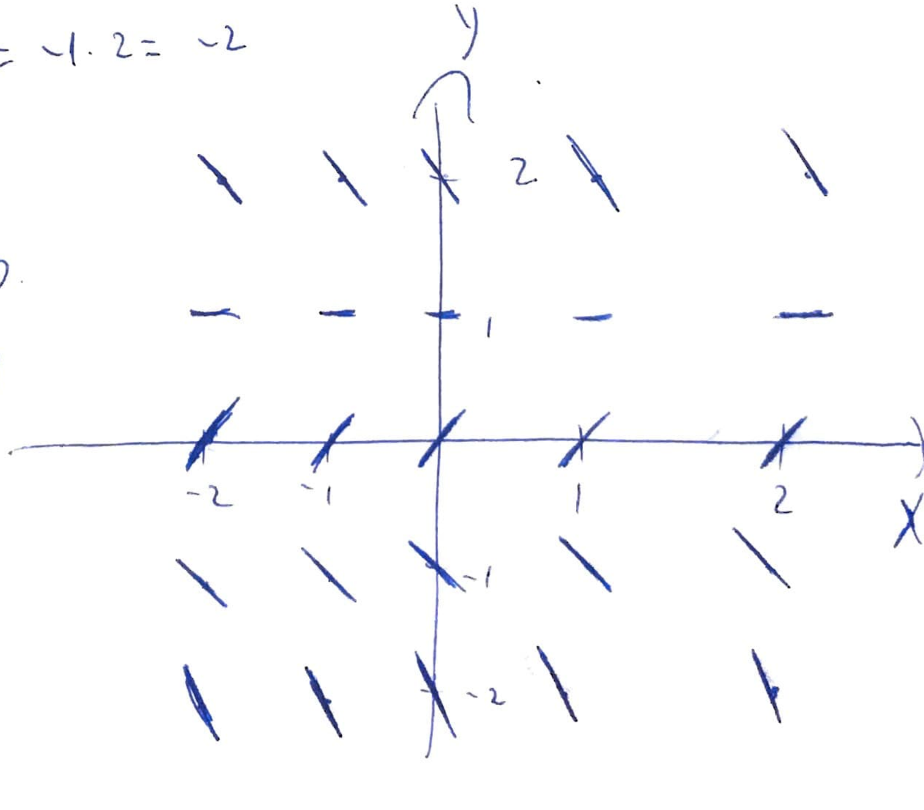
at $y=-2 \iff -2\cdot(1-(-2)) = -2\cdot(1+2) = -6$

at $y=-1 \iff -1\cdot(1-(-1)) = -1\cdot 2 = -2$

at $y=0 \iff 0\cdot(1-0) = 1$

at $y=1 \iff 1\cdot(1-1) = 0$

at $y=2 \iff 1\cdot(1-2) = -1$



Extra :

Solve $(x^2+1)y' + 4xy - x = 0$ $\Leftrightarrow$

$$y' + \frac{4x}{x^2+1} y = \frac{x}{x^2+1}$$

First we solve $\frac{dy}{dx} + \frac{4x}{x^2+1} y = 0$ $\Leftrightarrow$ assume $y \neq 0$.

$$\frac{dy}{y} = \frac{-4x}{x^2+1} dx \Leftrightarrow \text{integrate and get } \int \frac{dy}{y} = \int \frac{-4x}{x^2+1} dx \Leftrightarrow$$

$$\ln|y| = -2\ln(x^2+1) + \ln c_1 \text{ , } c_1 > 0 \Leftrightarrow \ln|y| = \ln\left|\frac{c_1}{(x^2+1)^2}\right| \Leftrightarrow$$

$$|y| = \frac{c_1}{(x^2+1)^2} \Leftrightarrow \boxed{y = \frac{c}{(x^2+1)^2}} \text{ , } c = \pm c_1 \neq 0$$

then we search the solution on the form $\boxed{y = \frac{\alpha(x)}{(x^2+1)^2}}$ $\Leftrightarrow$

we have $y' + \frac{4x}{x^2+1} \cdot y = \frac{x}{x^2+1}$ $\Leftrightarrow$.

$$\alpha'(x) \cdot \frac{1}{(x^2+1)^2} - \cancel{\alpha(x) \cdot \frac{2(x^2+1)2x}{(x^2+1)^4}} + \cancel{4x\,\alpha(x) \cdot \frac{1}{(x^2+1)^2}} = \frac{x}{x^2+1}$$

$$\Leftrightarrow \alpha'(x) \cdot \frac{1}{(x^2+1)^2} = \frac{x}{x^2+1} \Leftrightarrow \alpha'(x) = x^3 + x \Leftrightarrow$$

$$\alpha(x) = \frac{x^4}{4} + \frac{x^2}{2} + C \Leftrightarrow$$

$$\boxed{y = \left(\frac{x^4}{4} + \frac{x^2}{2} + C\right) \cdot \frac{1}{(x^2+1)^2}}$$