Worksheet 27:

Exercise 1:

(a) $y = \sin(3x) + 2e^{4x}$ a solution of $y'' + 9y = 50e^{4x}$?

We need to check:

$$y'' = -9\sin(3x) + 2\cdot 4\cdot 4\, e^{4x} = -9\sin(3x) + 32e^{4x} \quad (\in)$$

$$y'' + 9y = -9\sin(3x) + 32e^{4x} + 9\sin(3x) + 18e^{4x} = 50e^{4x}$$

(b) Since $y' = y^2 + 6$ is always positive, any solution function y must be always increasing.

(c) A differential equation is linear if it is of the form $a_n(x)\, y^{(n)} + \dots + a_1(x)\, y' + a_0(x)\, y = b(x)$, and is nonlinear otherwise.

Exercise 2:

$y = e^{rx}$, $\quad y' = re^{rx}$, $\quad y'' = r^2 e^{rx} \quad (\in)$

$$y'' + y' - 12y = 0$$

$$r^2 e^{rx} + re^{rx} - 12e^{rx} = 0$$

$$r^2 + r - 12 = 0$$

$$(r+4)(r-3) = 0 \rightarrow \boxed{r = -4, 3}$$

Exercise 4:

Tank of 200 liters of salt water solution

salt concentration of 2 grams/liters pours into it at a rate of 4 liters/minute. It also pours out out at 4 liters/minute.

Set up the Differential equation.

$$\frac{dy}{dt} = \text{rate}_{in} - \text{rate}_{out}.$$

rate in = rate of salt entering the tank.

$$= \underbrace{(2 \text{ grams/liter})}_{\text{concentration}} \cdot \underbrace{(4 \text{ liters/minute})}_{\text{sol. rate in}} = 8 \text{ g/min}$$

rate out = rate of salt leaving the tank.

$$= \text{???} \cdot (4 \text{ liters/minute})$$

$$\Rightarrow \frac{\text{mass of salt}}{\text{vol. of solution}} = \frac{y(t)}{200 - 0 \cdot t} \quad 4-4 \quad = \frac{y}{200} \Rightarrow$$

DE is $\boxed{\dfrac{dy}{dt} = 8 - \dfrac{y}{200}}$

Exercise 3:

Set up DE:

$$\frac{dy}{dt} = \text{rate}_{in} - \text{rate}_{out}$$

$\text{rate}_{in} = 0.1$

0.1 kg/min

$\text{rate}_{in} = 0.1$ ~~kg/liter · 10 liters/min~~

~~= 1 kg/min = 1000 gr/min.~~

⇒ pouring in

$$\frac{0.1}{10} = 0.01 \text{ kg/liter}$$

Concentration of salt in the tank at time t is

$$\frac{\text{mass of salt}}{\text{vol. of solution}} = \frac{y+10}{100}$$

$$\frac{dy}{dt} = 0.1 - \frac{y}{100}$$

Extra:

Solve $y' = 4y + 24$ subject to the condition $y(0) = 5$.

$\frac{dy}{dx} = 4y + 24. \Rightarrow \frac{dy}{dx} = 4(y+6) \Leftrightarrow$

$\frac{1}{y+6} \, dy = 4 \, dx \Leftrightarrow \int \frac{dy}{y+6} = \int 4 \, dx \Leftrightarrow$

$\ln|y+6| = 4x + k \Leftrightarrow$

$y+6 = Ce^{4x} \Leftrightarrow \boxed{y = -6 + Ce^{4x}}$

Use initial condition: $y(0) = 5$

$5 = -6 + C \Leftrightarrow \boxed{C = 11} \Rightarrow \boxed{y = -6 + 11e^{4x}}$

Extra:

Use separation of variables to find the general solutions:

a) $y' + 4xy^2 = 0 \iff 4xy^2 = -y' \iff$

$$4x = \frac{-1}{y^2}\frac{dy}{dx}$$

$$4x\,dx = -\frac{1}{y^2}\,dy \iff \int 4x\,dx = \int -\frac{1}{y^2}\,dy$$

$$\Rightarrow 2x^2 + c = \frac{1}{y} \Rightarrow \boxed{y = \frac{1}{2x^2+c}}$$

(b) $\sqrt{1-x^2}\,y' = xy \iff \sqrt{1-x^2}\,\frac{dy}{dx} = xy$

$$u = 1-x^2$$

$$\int \frac{1}{y}\,dy = \int \frac{x\,dx}{\sqrt{1-x^2}} \qquad du = -2x\,dx$$

$$-\frac{1}{2}\,du = x\,dx$$

$$\Rightarrow \ln|y| = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\left(2\sqrt{1-x^2}\right) + c = -\sqrt{1-x^2} + c$$

$$y = e^{-\sqrt{1-x^2}+c} = \boxed{C\cdot e^{-\sqrt{1-x^2}} = y}$$