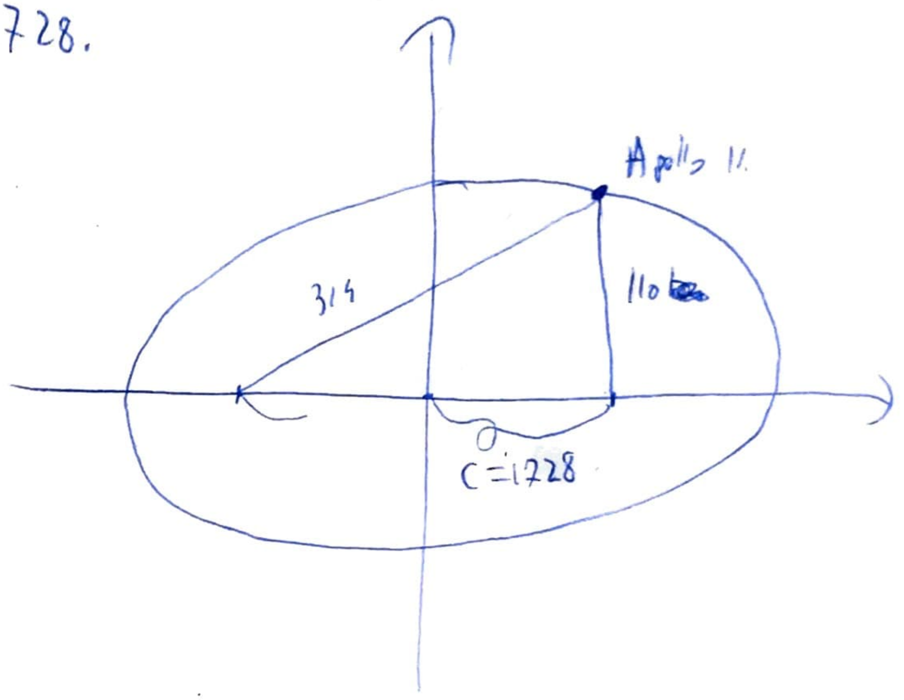
Worksheet 26:

Exercise 1: Apollo 11 spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km. Radius of moon is 1728.



$314 + 110 = 424 = 2a \Rightarrow$

$\boxed{a = 212}$

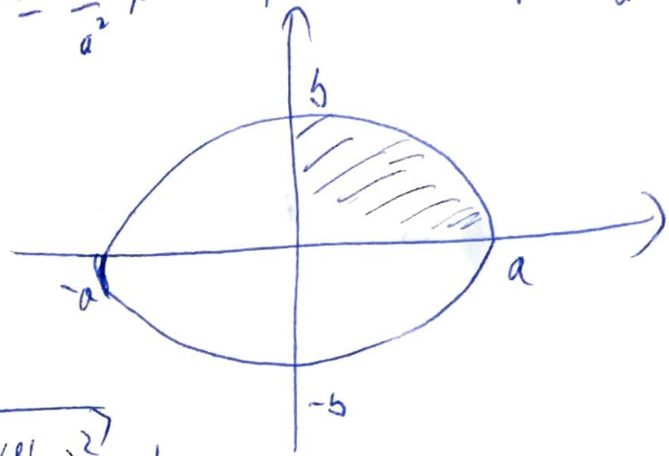
$a > b > 0$

$\Rightarrow c^2 = a^2 - b^2 \Rightarrow$

b^2

Exercise 7:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \Rightarrow y = \sqrt{b^2 - \frac{b^2}{a^2}x^2} = f(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$



$$S = 2\pi \cdot 2 \cdot \int_0^a f(x) \cdot \sqrt{1 + (f'(x))^2}\, dx.$$

$$f'(x) = \left(b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}\right)' = b \cdot \frac{1}{2} \cdot \left(1 - \frac{x^2}{a^2}\right)^{-\frac{1}{2}} \cdot \left(\frac{-2x}{a^2}\right) =$$

$$= \boxed{-\frac{bx}{a^2} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} = f'(x)}$$

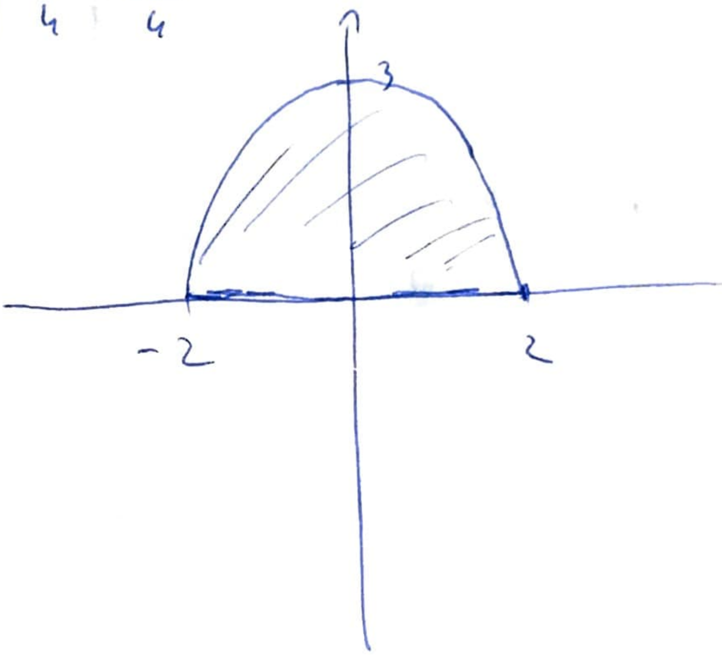
$$(f'(x))^2 = \frac{b^2x^2}{a^4} \cdot \frac{1}{1 - \frac{x^2}{a^2}} = \frac{b^2x^2}{a^4} \cdot \frac{1}{\frac{a^2 - x^2}{a^2}} =$$

$$= \frac{b^2x^2}{\cancel{a^4}^{\,a^2}} \cdot \frac{\cancel{a^2}}{a^2 - x^2} = \frac{b^2x^2}{a^4 - a^2x^2} = (f'(x))^2.$$

Exercise 6:

Find centroid of upper half of ellipse $9x^2+4y^2=36$.

$$4y^2=36-9x^2 \Rightarrow y=\sqrt{\frac{36}{4}-\frac{9x^2}{4}}$$



$$M_y=\rho\int_{-2}^{2} x\cdot\sqrt{9-\frac{9x^2}{4}}\,dx$$

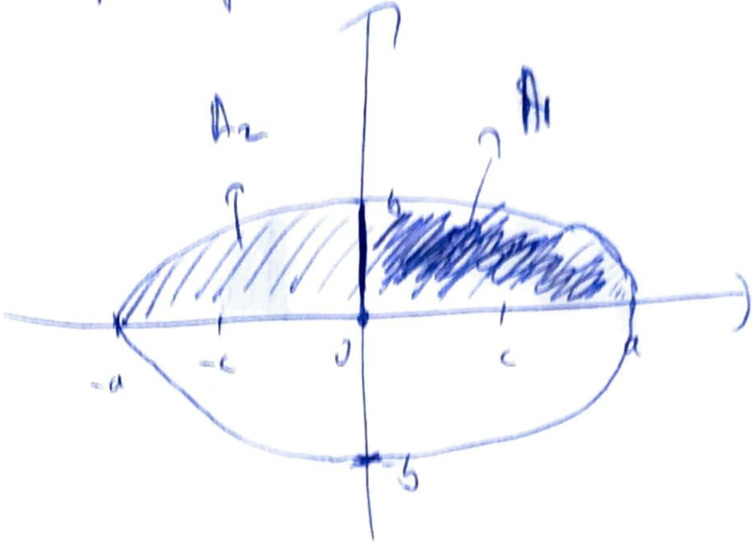
$$M_x=\frac{\rho}{2}\int_{-2}^{2}\left(9-\frac{9x^2}{4}\right)dx.$$

$$M=\rho\cdot A=\rho\cdot\int_{-2}^{2}\sqrt{9-\frac{9x^2}{4}}\,dx.$$

$$x_{cm}=\frac{M_y}{M}\quad,\quad y_{cm}=\frac{M_x}{M}.$$

Exercise 5 :

Find the volume of an ellipse, if it is rotated about its major axis.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

rotating about x-axis.

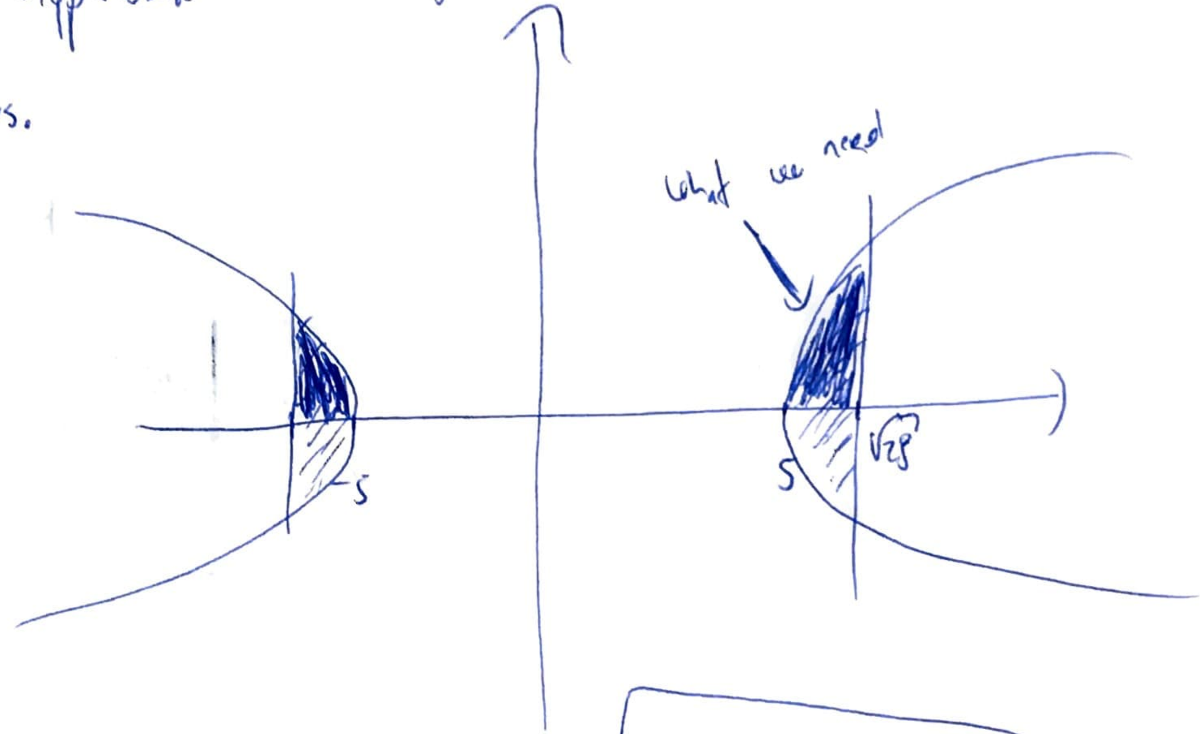
$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2 - \frac{b^2 x^2}{a^2} \Rightarrow y = \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}$$

Using disks $\Rightarrow$

$$V = 2\int_0^a \pi \cdot \left(b^2\left(1 - \frac{x^2}{a^2}\right)\right) dx = \boxed{2\pi b^2 \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = V}$$

Exercise 4 :

Area of the hyperbola $4x^2 - 25y^2 = 100$ and vertical line through a focus.



$$c = \sqrt{a^2 + b^2}$$

$$\frac{4x^2}{100} - \frac{25y^2}{100} = 1 \iff \boxed{\frac{x^2}{25} - \frac{y^2}{4} = 1}$$

$$c = \sqrt{25+4} = \sqrt{29} \approx \boxed{5,385}$$

$$\frac{y^2}{4} = \frac{x^2}{25} - 1 \Rightarrow y^2 = \frac{4x^2}{25} - 4 \Rightarrow$$

$$\boxed{y = \sqrt{\frac{4x^2}{25} - 4}}$$

$$\boxed{A = \int_5^{\sqrt{29}} \sqrt{\left(\frac{2x}{5}\right)^2 - 2^2}\, dx}$$

$$\boxed{\text{Substitute } \frac{2x}{5} = 2\sec(t).}$$

$x'(t) = -2\sin(t)$

$$\Rightarrow \left(x'(t)\right)^2 + \left(y'(t)\right)^2 = 4\sin^2(t) + 9\cos^2(t)$$

$y'(t) = 3\cos(t)$

$$\Rightarrow \boxed{C = \int_0^{2\pi} \sqrt{4\sin^2(t) + 9\cos^2(t)}\, dt}$$

$$\Delta x = \frac{2\pi}{12} = \frac{\pi}{6}.$$

Simpson's $n = 12$

t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}
								$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$		
0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π

$$f(t) = \sqrt{4\sin^2(t) + 9\cos^2(t)}.$$

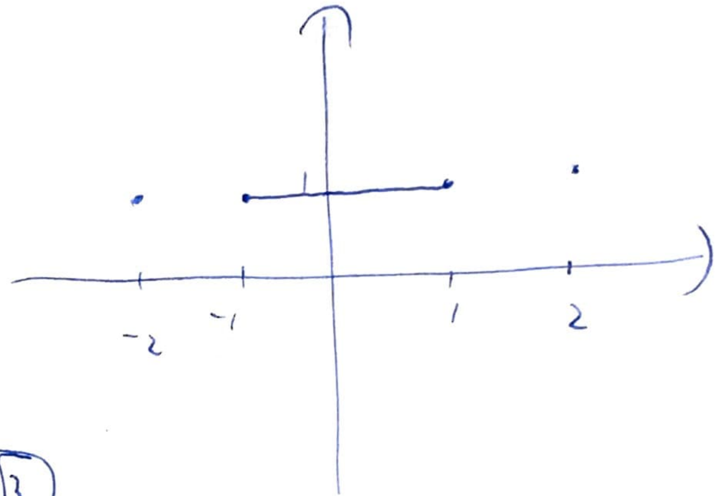
$$\int_0^{2\pi} \sqrt{4\sin^2(t) + 9\cos^2(t)}\, dt = \frac{\pi}{18}\Big(f(t_0) + 4f(t_1) + 2f(t_2) + 4f(t_3) + 2f(t_4)$$

$$+ 4f(t_5) + 2f(t_6) + 4f(t_7) + 2f(t_8) + 4f(t_9) +$$

$$+ 2f(t_{10}) + 4f(t_{11}) + f(t_{12}) \Big).$$

Exercise 2 :

Equation of an ellipse with foci (1, 1) and (−1, 1) and major axis of length 4.



$c = 1$ $\quad a > b > 0$

$a = 2$

$c^2 = a^2 - b^2$

$b^2 = a^2 - c^2 = 4 - 1 = 3 \Rightarrow \boxed{b = \sqrt{3}} \Rightarrow$

$$\boxed{\frac{(x-0)^2}{4} + \frac{(y-1)^2}{3} = 1}$$

Exercise 3 : $9x^2 + 4y^2 = 36$ using Sympson's rule with $n = 12$.

we have $\frac{9x^2}{36} + \frac{4y^2}{36} = 1 \Leftrightarrow \frac{x^2}{4} + \frac{y^2}{9} \Leftrightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{x}{2} = \cos t \Rightarrow \boxed{x = 2\cos(t)}$ $\qquad \frac{y}{3} = \sin(t) \Rightarrow \boxed{y = 3\sin(t)}$

$$\begin{cases} x = 2\cos(t) \\ y = 3\sin(t) \end{cases} \quad 0 \le t \le 2\pi.$$

$$\text{Circumference} = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$f(x)\cdot\sqrt{1+\left(f'(x)\right)^2} = b\cdot\sqrt{1-\frac{x^2}{a^2}}\cdot\sqrt{1+\frac{b^2x^2}{a^4-a^2x^2}} =$$

$$= b\cdot\sqrt{\frac{a^2-x^2}{a^2}}\;\sqrt{\frac{a^2(a^2-x^2)+b^2x^2}{a^2(a^2-x^2)}} =$$

$$= \frac{b}{a}\cancel{\sqrt{a^2-x^2}}\cdot\frac{1}{a}\cdot\frac{\sqrt{a^4-a^2x^2+b^2x^2}}{\cancel{\sqrt{a^2-x^2}}} =$$

$$= \boxed{\frac{b}{a^2}\cdot\sqrt{a^4-(a^2-b^2)x^2} = f(x)\sqrt{1+\left(f'(x)\right)^2}}$$

$$S = 4\pi\cdot\frac{b}{a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2}\,dx$$

substitute $x = \frac{a^2}{\sqrt{a^2-b^2}}\cdot\sin(t)$ and continue.