Worksheet 25:

Exercise 1:

(a) $r = 2\cos\theta + 1 \Rightarrow$ $x = r\cos\theta = 2\cos^2\theta + \cos\theta$

$y = r\sin\theta = 2\cos\theta\sin\theta + \sin\theta$

$\boxed{x' = 4\cos(\theta)\cdot(-\sin\theta) - \sin\theta}$ $\quad y' = 2\cos\theta\cos\theta - 2\sin\theta\sin\theta + \cos\theta.$

$= 2\cos^2\theta - 2\sin^2\theta + \cos\theta.$

$$\frac{dy}{dx} = \frac{2\cos^2\theta - 2\sin^2\theta + \cos\theta}{-4\cos\theta\sin\theta - \sin\theta}$$

(b) $r = \frac{1}{\theta} \Rightarrow x = \frac{1}{\theta}\cos\theta$, $y = \frac{1}{\theta}\sin\theta \Rightarrow$

$$x' = \frac{-\theta\sin\theta - \cos(\theta)\cdot(1)}{\theta^2} = \frac{-\theta\sin\theta - \cos\theta}{\theta^2}, \quad y' = \frac{\theta\cos\theta - \sin\theta}{\theta^2}.$$

$$\frac{dy}{dx} = \frac{\theta\cos\theta - \sin\theta}{\cancel{\theta^2}} \cdot \frac{\cancel{\theta^2}}{-\theta\sin\theta - \cos\theta} = \frac{\theta\cos\theta - \sin\theta}{-\theta\sin\theta - \cos\theta}$$

(c) $r = 2e^{-\theta}$ $\quad x = 2e^{-\theta}\cos\theta$, $y = 2e^{-\theta}\sin\theta.$

$x' = 2e^{-\theta}(-\sin\theta) - 2e^{-\theta}\cos\theta$, $\quad y' = 2e^{-\theta}\cos\theta - 2e^{-\theta}\sin\theta.$

$$\frac{dy}{dx} = \frac{2e^{-\theta}\cos\theta - 2e^{-\theta}\sin\theta}{-2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta} = \frac{\cos\theta - \sin\theta}{-\sin\theta - \cos\theta}.$$

Exercise 2:

(a) $r = \sin\theta$ $\quad \theta = \frac{\pi}{3}$ $\quad x = \sin\theta\cos\theta$

$y = \sin^2\theta$

$x'(\theta) = -\sin^2\theta + \cos^2\theta$, $y'(\theta) = 2\sin\theta\cdot\cos\theta$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{2\sin\theta\cdot\cos\theta}{\cos^2\theta - \sin^2\theta} \Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{2\left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2} = -\sqrt{3}$$

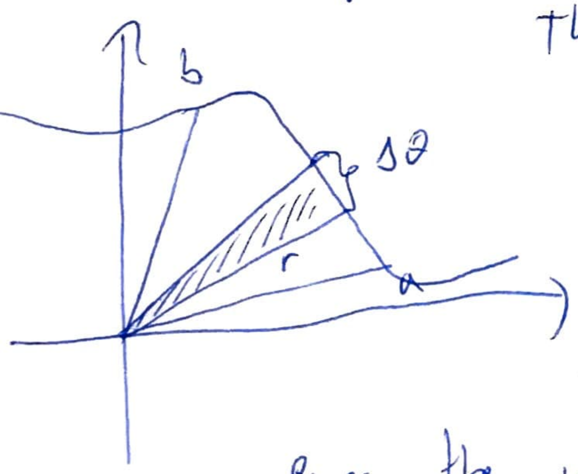
(b) $r = \frac{1}{\theta}$, $x = \frac{1}{\theta}\cos\theta$, $y = \frac{1}{\theta}\sin\theta$ at $\theta = \frac{\pi}{2}$

$x'(\theta) = -\frac{1}{\theta^2}\cos\theta - \frac{1}{\theta}\sin\theta$, $y'(\theta) = -\frac{1}{\theta^2}\sin\theta + \frac{1}{\theta}\cos\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{-\left(\frac{2}{\pi}\right)^2}{-\frac{2}{\pi}} = \frac{2}{\pi}$$

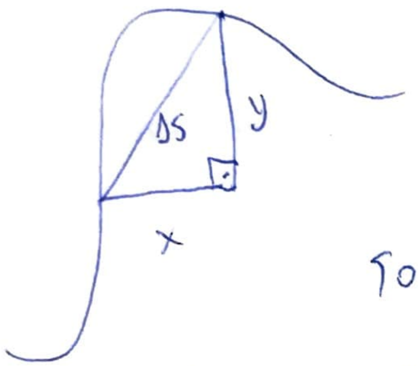
Exercise 3:

(a) $\text{Area} = \frac{1}{2}\int_a^b r^2\,d\theta = \frac{1}{2}\int_a^b f(\theta)^2\,d\theta$



The area of a slice is calculated approximately as the area of a circular sector of radius r: $\frac{1}{2}r^2\Delta\theta$. the limit of the Riemann sum gives the integral.

(b) Arclength $= \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$



We have $r = f(\theta) = \frac{x}{\cos\theta} = \frac{y}{\sin\theta}$,

So $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$.

Then recall arc length is given by

$$\int_a^b \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta = \dots = \int_a^b \sqrt{f(\theta)^2 + (f'(\theta))^2}\, d\theta$$

(c) A circle of (constant) radius r has area

$$\frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\left[\theta r^2\right]\Big|_0^{2\pi} = \pi r^2$$ and circumference.

(thinking $f(\theta) = r$ so $f'(\theta) = 0$) $\Rightarrow$

$$\int_0^{2\pi} \sqrt{r^2 + 0^2}\, d\theta = \int_0^{2\pi} r\, d\theta = r\theta\Big|_0^{2\pi} = 2\pi r.$$

Exercise 4:

$r = \theta^2$ at $\theta = \pi$.

$x = r\cos\theta = \theta^2 \cos\theta$, $y = r\sin\theta = \theta^2 \sin\theta$.

$x' = -\theta^2 \sin\theta + 2\theta\cos\theta \Rightarrow x'(\pi) = -\pi^2 0 - 2\pi = -2\pi$.

$y' = \theta^2\cos\theta + 2\theta\sin\theta \Rightarrow y'(\pi) = \pi^2(-1) + 2\pi . 0 = -\pi^2$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \left.\frac{y'(\theta)}{x'(\theta)}\right|_{\theta=\pi} = \frac{-\pi^2}{-2\pi} = \frac{\pi}{2}.$$

Exercise 5:

$r = 2 + \sin\theta$

$x = r\cos\theta = (2+\sin\theta)\cos\theta = 2\cos\theta + \sin\theta\cos\theta$

$x' = -2\cos\theta - \sin^2\theta + \cos^2\theta$

$y = r\sin\theta = (2+\sin\theta)\sin\theta = 2\sin\theta + \sin^2\theta \Rightarrow$

$y' = 2\cos\theta + 2\sin\theta \cdot \cos\theta$.

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{2\cos\theta + 2\sin\theta\cdot\cos\theta}{-2\sin\theta - \sin^2\theta + \cos^2\theta} = \frac{2\cos\theta(1+\sin\theta)}{\cos^2\theta - 2\sin\theta - \sin^2\theta}$$

$\frac{dy}{dx} = 0$ when $\cos\theta = 0$ or $\sin\theta = -1$, thus when

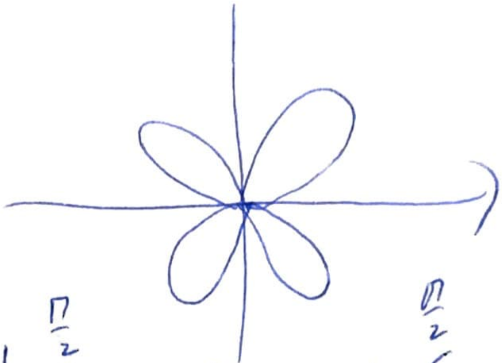
$\theta = \frac{\pi}{2} + 2\pi n$ or $\theta = \frac{3\pi}{2} + 2\pi n$ (or more simply, $\theta = \frac{\pi}{2} + 2\pi n$)

One can quickly verify that $\cos^2\frac{\pi}{2} - 2\sin\frac{\pi}{2} - \sin\frac{\pi}{2} = -3 \neq 0$

and $\cos^2\left(\frac{3\pi}{2}\right) - 2\sin\left(\frac{3\pi}{2}\right) - \sin^2\left(\frac{3\pi}{2}\right) = 1 \neq 0$.

Exercise 6 :

$r = \sin(2\theta)$



$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\, d\theta =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2}\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1}{2} - \frac{\cos(4\theta)}{2}\right] d\theta = \frac{1}{2}\left(\frac{\theta}{2}\right)\Big|_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{2\pi} \frac{\cos u}{8}\, du$$

$$= \frac{1}{2}\left(\frac{\theta}{2}\right)\Big|_0^{\frac{\pi}{2}} - \frac{1}{2}\,\frac{\sin u}{8}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{8}.$$

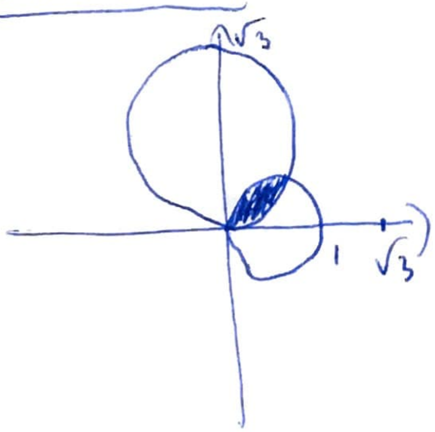
$$\boxed{\begin{aligned} u &= 4\theta \\ du &= 4\, d\theta \\ \frac{du}{4} &= d\theta. \end{aligned}}$$

Exercise 8 :

$r = \cos(\theta) \qquad 0 \leq \theta \leq \frac{\pi}{4}.$

$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{\pi}{16} + \frac{1}{8}.$$

Exercise 9 :



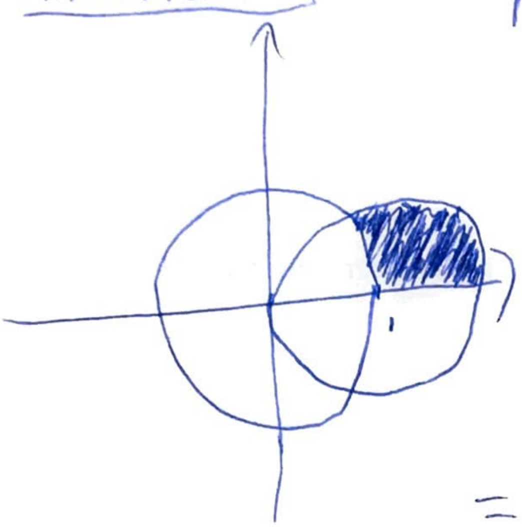
$$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} (\sqrt{3}\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos\theta)^2\, d\theta =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} 3\left(\frac{1-\cos 2\theta}{2}\right) d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta =$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - \frac{3\sin 2\theta}{4}\right]\Big|_0^{\frac{\pi}{6}} + \frac{1}{2}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]\Big|_{\frac{\pi}{6}}^{\pi/2} =$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right) + \frac{\pi}{8} - \frac{1}{2}\left[\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right].$$

Exercise 10:



$$A = \frac{1}{2}\int_0^{\pi/3} (4\cos^2\theta - 1)\, d\theta =$$

$$= \frac{1}{2}\int_0^{\pi/3} \left(\frac{4+4\cos 2\theta}{2} - 1\right) d\theta =$$

$$= \frac{1}{2}\int_0^{\pi/3} (1 + 2\cos(2\theta)\, d\theta - \frac{1}{2}\left[\theta + \frac{2\sin 2\theta}{2}\right]\Big|_0^{\pi/3} =$$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$$= \frac{1}{2}\left[\theta + \sin 2\theta\right]\Big|_0^{\pi/3} = \frac{1}{2}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right] = \frac{\pi}{6} + \frac{\sqrt{3}}{4}.$$

Exercise 11: length of the curve $r = \theta^2$ for $0 \le \theta \le 2\pi$

$$r' = 2\theta,\ \Rightarrow\ S = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2} = \int_0^{2\pi} \sqrt{\theta^2(\theta^2+4)}\, d\theta =$$

$$= \int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta =$$

$u = \theta^2 + 4$

$du = 2\theta\, d\theta$

$\frac{du}{2} = \theta\, d\theta$

$$= \int_4^{4\pi^2+4} \frac{1}{2}\sqrt{u} = \frac{1}{2}\cdot\frac{2}{3} u^{\frac{3}{2}}\Big|_4^{4\pi^2+4} = \frac{1}{3}(\theta^2+4)^{\frac{3}{2}}\Big|_0^{2\pi} = \frac{1}{3}(4\pi^2+4)^{3/2} - \frac{1}{3}4^{3/2}$$

Exercise 12:

$r = \sin\theta + \theta \Rightarrow r' = \cos\theta + 1$

$$s = \int_0^{\pi} \sqrt{(\sin\theta + \theta)^2 + (\cos(\theta)+1)^2}\, d\theta.$$