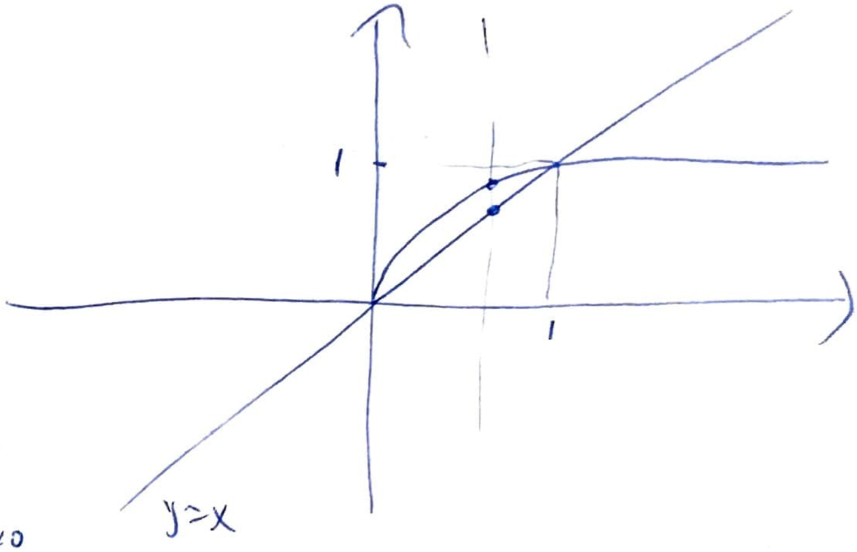
Worksheet 24:

Exercise 4:

$y = \sqrt{x}$ and $y = x$.



first we need to find the point of intersection of two graphs.

$$x = \sqrt{x} \iff x^2 = x \iff x^2 - x = 0 \iff x(x-1) = 0 \iff \boxed{\begin{matrix} x=0 \\ x=1 \end{matrix}}$$

$$M_y = \rho \int_0^1 x \cdot (\sqrt{x} - x)\, dx = \rho \int_0^1 (x \cdot x^{\frac{1}{2}} - x^2)\, dx =$$

$$= \rho \left[\int_0^1 x^{3/2} dx - \int_0^1 x^2 dx \right] = \rho \left[\frac{2}{5} \cdot x^{\frac{5}{2}} \Big|_0^1 - \frac{x^3}{3} \Big|_0^1 \right] =$$

$$= \rho \left[\frac{2}{5} - \frac{1}{3} \right] = \rho \left[\frac{6 - 5}{15} \right] = \rho \frac{1}{15}$$

$$M_x = \frac{1}{2} \rho \int_0^1 (x - x^2)\, dx \quad , \quad M = \rho \cdot A = \rho \cdot \int_0^1 (\sqrt{x} - x)\, dx.$$

$$x_{cm} = \frac{M_y}{m} \quad , \quad y_{cm} = \frac{M_x}{M}.$$

Exercise from quiz:

Find the arc length of the curve

$$x = e^t - t \quad , \quad y = 4e^{\frac{t}{2}} \quad , \quad 0 \le t \le 2.$$

Solution:

Using
$$\boxed{A^2 + 2AB + B^2 = (A+B)^2}$$

$x'(t) = e^t$, $y'(t) = 2e^{\frac{t}{2}}$ and so,

$$[x'(t)]^2 + [y'(t)]^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2$$

We obtain:

$$L = \int_0^2 \sqrt{\{x'(t)\}^2 + (y'(t))^2}\, dt = \int_0^2 \sqrt{(e^t + 1)^2}\, dt =$$

$$= \int_0^2 (e^t + 1)\, dt = e^2 + 1$$

Exercise 5:

$y = 3\sin(x) + \cos(2x)$ on $[0, \pi]$ $\Rightarrow$ Average function.

~~Avg~~ $\text{Avg} = \frac{1}{\pi - 0} \int_0^{\pi} \left(3\sin(x) + \cos(2x)\right) dx =$

$= \frac{1}{\pi} \left[3 \int_0^{\pi} \sin(x)\, dx + \int_0^{\pi} \cos(2x)\, dx \right]$

Exercise 3:

(a) $x = a\cos^3(\theta)$, $y = a\sin^3(\theta)$ $0 \le \theta \le 2\pi$.

$x'(\theta) = -3a\cos^2(\theta)\sin(\theta)$ $y'(\theta) = 3a\sin^2(\theta)\cos(\theta)$

$(x'(\theta))^2 = 9a^2\cos^4(\theta)\sin^2(\theta)$ $(y'(\theta))^2 = 9a^2\sin^4(\theta)\cos^2(\theta)$

$L = \int_0^2 \sqrt{9a^2\cos^2(\theta)\sin^2(\theta)\underbrace{\left(\cos^2(\theta) + \sin^2(\theta)\right)}_{=1}}\, d\theta =$

$= \int_0^2 3a\cos(\theta)\sin(\theta)\, d\theta.$

(b) $y = \sqrt{2-x^2}$, $0 \le x \le 1$.

$$y' = \frac{1}{2}(2-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{2-x^2}} \Rightarrow (y')^2 = \frac{x^2}{2-x^2}$$

$$L = \int_0^1 \sqrt{1 + \frac{x^2}{2-x^2}}\, dx = \int_0^1 \sqrt{\frac{2-x^2+x^2}{2-x^2}}\, dx =$$

$$= \int_0^1 \frac{\sqrt{2}}{\sqrt{2-x^2}}\, dx = \sqrt{2} \int_0^1 \frac{dx}{\sqrt{2-x^2}} .$$

Substitute $x = \sqrt{2} \cdot \sin(u)$ and continue.

Exercise 2:

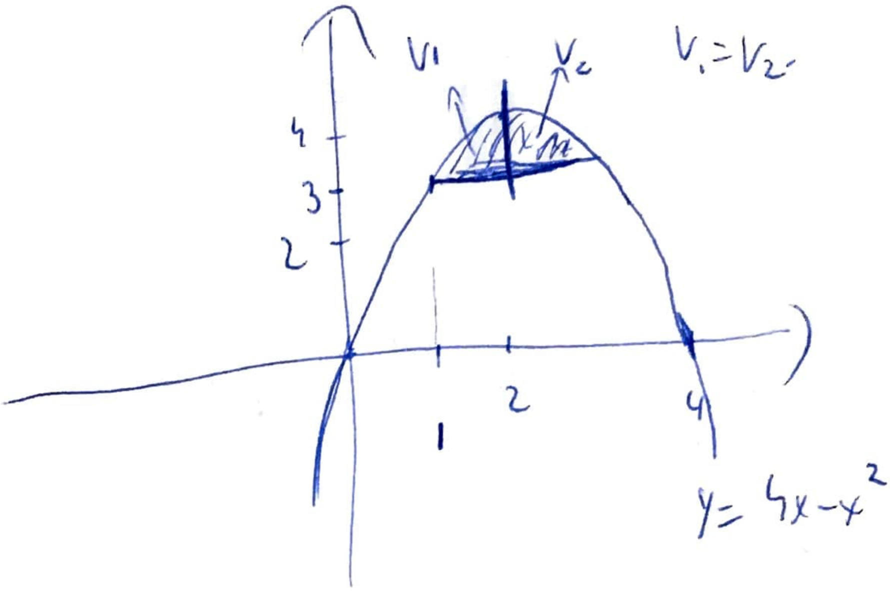
(a) $y = \sqrt{x+1}$, $0 \le x \le 3$ about y-axis.

$$y' = \frac{1}{2\sqrt{x+1}} \Rightarrow (y')^2 = \frac{1}{4x+4} \Rightarrow S = 2\pi \int_0^3 \sqrt{x+1} \cdot \sqrt{1 + \frac{1}{4x+4}}\, dx =$$

$$= 2\pi \int_0^3 \sqrt{x+1} \cdot \sqrt{\frac{4x+5}{4(x+1)}}\, dx = 2\pi \int_0^3 \cancel{\sqrt{x+1}} \cdot \frac{\sqrt{4x+5}}{2 \cdot \cancel{\sqrt{x+1}}}\, dx =$$

$$= \pi \int_0^3 \sqrt{4x+5}\, dx$$, substitute $4x+5 = u$ and continue.

(c) The solid: $y = 4x - x^2$ and $y = 3$ about the line $x = 1$.



Since we are rotating about $x = 1$ we need $x = f(y)$ ↳ Almost oy.

$$\Rightarrow -y = x^2 - 4x \iff$$

$$-y = x^2 - 4x + 4 - 4 \Rightarrow 4 - y = (x-2)^2 \iff \pm\sqrt{4-y} = x - 2$$

$$x = 2 \pm \sqrt{4-y}$$

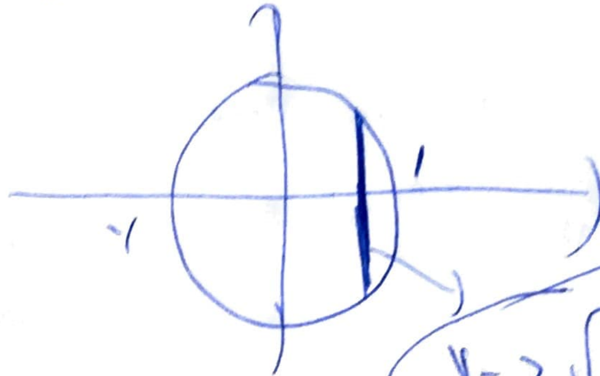
we take only $x = 2 + \sqrt{4-y}$ and multiply the volume by 2 since it is the same.

$$A = \pi\cdot\left(2+\sqrt{4-y}-1\right)^2 - \pi(2-1)^2 = \pi\cdot\left(1+\sqrt{4-y}\right)^2 - \pi$$

$$V = 2\pi\int_3^4 \left(\left(1+\sqrt{4-y}\right)^2 - 1\right) dy$$

(d) The solid with circular base of radius 1 and cross sections perpendicular to the base are equilateral triangles.

$x^2 + y^2$.



Area of equilateral triangles is $\boxed{A = \frac{\sqrt{3}}{4} a^2}$

$y = 2\sqrt{1-x^2}$

One side of our triangle

Area of cross section $= \frac{\sqrt{3}}{4} \cdot \left(2\sqrt{1-x^2}\right)^2 = \sqrt{3}(1-x^2)$

$$\boxed{V = \int_{-1}^{1} \sqrt{3}(1-x^2)\,dx.}$$

Extra:

Find the parametric equation of a line who passes through $(\underset{x_1}{3}, \underset{y_1}{5})$ and $(\underset{x_2}{5}, \underset{y_2}{9})$.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-5}{5-3} = \frac{4}{2} =$$

$= b$

$= a.$

Pick any of the points.

$(\underset{x_0}{3}, \underset{y_0}{5})$

$x = x_0 + at$

$y = y_0 + bt$

$$\boxed{\begin{aligned} x &= 3 + 2t \\ y &= 5 + 4t \end{aligned}}$$

$\overset{=a}{2}$ and $\overset{=b}{4}$ can change

but $\boxed{\frac{b}{a} = 2}$

Extra:

The parametric equations

$$x(t) = -2 + 4\cos(3t) \qquad y(t) = 2 - 4\sin(3t)$$

$$0 \le t \le \frac{2\pi}{3}.$$

$$\frac{x+2}{4} = \cos(3t) \qquad \frac{y-2}{4} = -\sin^3(t)$$

$$\frac{(x+2)^2}{4^2} + \frac{(y-2)^2}{4^2} = 1 = \cos^2(3t) + \sin^2(3t)$$

$$(x+2)^2 + (y-2)^2 = 4^2 \Rightarrow R = 4$$

center $(-2, 2)$

Extra:

The curve with parametric equations $x(t) = at + 3$ $y(t) = 2at^2 - 5$ contains $(1, -1)$. Find a?

$$at + 3 = 1 \Rightarrow at = 2 \ ; \ 2at^2 - 5 = -1 \Leftrightarrow$$

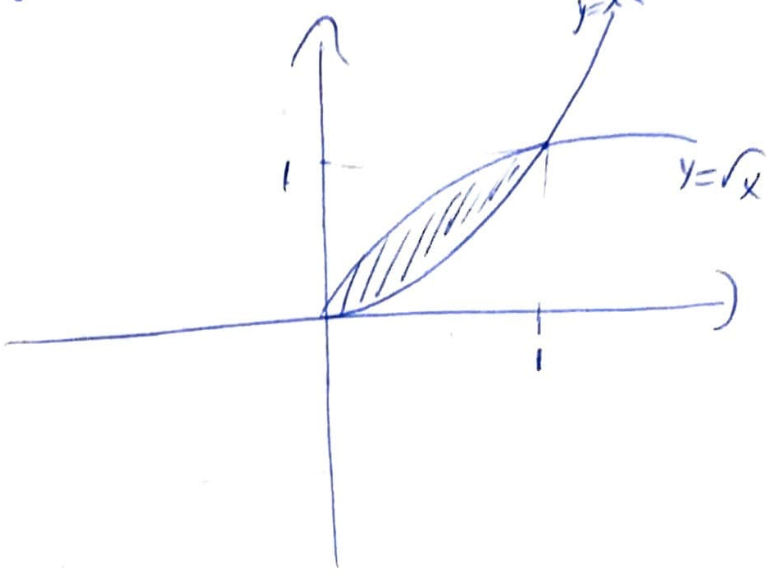
$$2(at)t - 5 = -1 \Rightarrow 2 \cdot 2 \cdot t = 4 \Rightarrow \boxed{t = 1}.$$

$$a = \frac{2}{t} = \frac{2}{1} = \boxed{1 = a}$$

Exercise 1:

(a) The solid ; $y=x^2$, $x=y^2$ about the x-axis.

One way of doing it is



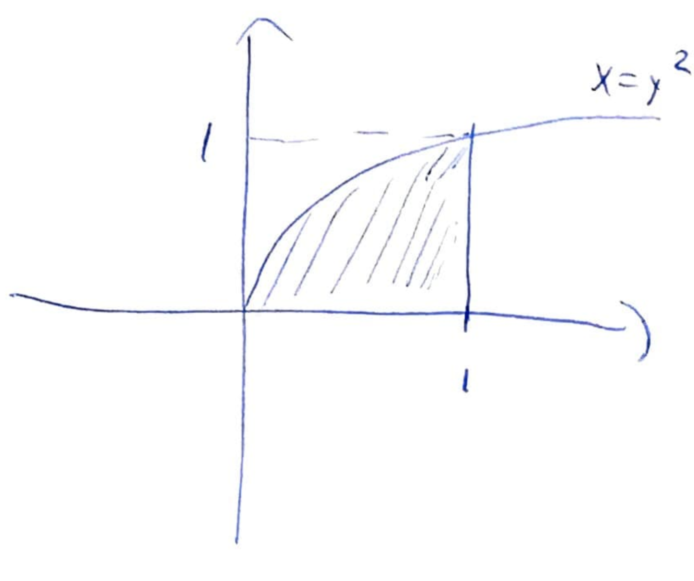
$$\text{Area} = \pi\cdot(\sqrt{x})^2 - \pi\cdot(x^2)^2$$

$$= \pi\cdot x - \pi\cdot x^4 = \pi(x-x^4)$$

$$\Rightarrow \boxed{V = \pi\int_0^1 (x-x^4)\,dx.}$$

(b) The solid ; $x=y^2$ and $x=1$ about the line $x=1$.

$\hookrightarrow y=\sqrt{x}$



$$A = \pi\cdot(1-y^2)^2 = \pi\cdot(1-2y^2+y^4)$$

$$V = \pi\int_0^1 (1-2y^2+y^4)\,dy$$

(b) $x = 3t^2 \quad y = 2t^3, \; 0 \le t \le 5$ about y-axis.

since rotating about y-axis

$$\text{Surface area} = 2\pi \int_0^5 x(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

$x'(t) = 6t, \quad y'(t) = 6t^2 \Rightarrow (x'(t))^2 = 36t^2, \quad (y'(t))^2 = 36t^4 \Rightarrow$

$$S = 2\pi \int_0^5 3t^2 \sqrt{36t^2 + 36t^4}\, dt = 2\pi \cdot 3 \cdot 6 \int_0^5 t^2 \cdot \sqrt{t^2(1+t^2)}\, dt =$$

$$= 2\pi \cdot 3 \cdot 6 \int_0^5 t^2 \cdot t \sqrt{1+t^2}\, dt. \qquad t^2 = u+1$$

First find $\int t^3 \sqrt{1+t^2}\, dt \Rightarrow$ let $1+t^2 = u \Rightarrow 2t\,dt = du$

$t\,dt = \frac{du}{2}$

$$I = \frac{1}{2} \int (u+1) \cdot \sqrt{u}\, du = \frac{1}{2} \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du =$$

$$= \frac{1}{2} \cdot \frac{2}{5} \cdot u^{\frac{5}{2}} + \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C = \boxed{\frac{1}{5}(1+t^2)^{\frac{5}{2}} + \frac{1}{3}(1+t^2)^{\frac{3}{2}} + C}$$

Continue.

Exercise 7; $(x(t), y(t)) = (t^2, 2t+1)$.

(a) Find tangent line at $(4,-3)$.

$t^2 = 4$ and $2t+1 = -3$

$\Downarrow \qquad\qquad \Downarrow$

$t = \pm 2 \qquad 2t = -4$

$\boxed{t = -2}$

$\frac{dx}{dt} = 2t \quad , \quad \frac{dy}{dt} = 2$

$x'(-2) = -4 \quad , \quad y'(-2) = 2.$

$\frac{dy}{dx} = \frac{x'(-2)}{y'(-2)} = \frac{-4}{2} = -2$

$\Rightarrow \boxed{y+3 = -2(x-4)}$

(b) Find $\frac{d^2y}{dx^2}$ at $(x,y) = (4,-3) \Rightarrow \boxed{t=-2}$

$\frac{d^2y}{dx^2} = \frac{x'(t)\,y''(t) - y'(t)\,x''(t)}{(x'(t))^3}$.

$x''(-2) = 2 \quad , \quad y''(-2) = 0$

$\frac{d^2(y)}{dx^2} = \frac{-4\cdot 0 - 2\cdot 2}{(-4)^3} = \frac{-4}{-4^3} = \frac{1}{16} > 0$

Concave up.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) =$

$= \frac{d}{dt}\cdot\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} =$

$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$