Worksheet 22:

Exercise 1:

(a) A parametrization of a curve is a map $\vec{r}(t) = \langle x(t), y(t) \rangle$ from a parameter $R = [a,b]$ to the plane. The functions $x(t)$, $y(t)$ are called coordinate functions. The image of a parametrization is called a parametrized curve in the plane. (Same in 3D). The parametrization contains more information about the curve than the curve alone, for example it tells how fast we go along the curve.

(b) $x' = 0$ means it is vertical, $y' > 0$ means it is moving upwards and $x'' > 0$ means that it is possibly concaving rightward.

c) parametric equations that represent the circle of radius 5 and center (2,4)

$$(x-2)^2 + (y+4)^2 = 5^2 = 25 \implies$$

Put $x - 2 = 5\cos(t)$ $\qquad y + 4 = 5\sin(t)$

$x = 5\cos(t) + 2$ $\qquad y = 5\sin(t) - 4.$

One should check that these satisfy the Cartesian equation.

(d) Represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{c^2} = 1$ with parametric equations

put $x = a\cos(t)$, $y = b\sin(t)$.

(e) $y_1(t) = 5\sin(t)$ $x_1(t) = 5\cos(t)$ $0 \le t \le 2\pi$

$y_2(t) = 5\sin(t)$ $x_2(t) = 5\cos(t)$, $0 \le t \le 2\pi$.

This is true, so they are equal in the sense that both have the same graph in the xy plane, of a circle of radius of radius 5. But for $0 \le t \le 2\pi$, the circle is traced out exactly once counterclockwise. For $0 \le t \le 20\pi$, the circle is traced at ten times.

Exercise 2:

Exercise 3:

(a) $x=\sqrt{t}$, $y=1-t$ $\Rightarrow$ $t=1-y$ $\Rightarrow$ substitute and get

$$\boxed{x=\sqrt{1-y}}$$

(b) $x=3t-5$, $y=2t+1$ $\Rightarrow$ $t=\frac{x+5}{3}$ $\Rightarrow$ substitute

$$y=2\cdot\frac{x+5}{3}+1 \Rightarrow 3y=2x+10+3=\boxed{2x+13=3y}$$

(c) $x=\cos(t)$, $y=\sin(t)$ $\Rightarrow$ $x^2+y^2=1$ since

$$\cos^2(t)+\sin^2(t)=1.$$

Exercise 4:

(a) $y=x^3$ from $x=0$ to $x=2$.

$$c(t)=(t,t^3) \text{ from } t=0 \text{ to } t=2.$$

(b) $\frac{x^2}{4}+\frac{y^2}{9}=1$ $\Rightarrow$ $c(t)=(2\cos(t),3\sin(t))$ for $0\le t\le 2\pi$.

Exercise 5:

t	x
0	2
5	-1

$$\frac{-1-2}{5-0}=\frac{-3}{5} \Rightarrow x=\frac{-3}{5}t+b$$

$$2=\frac{-3}{5}\cdot 0+b$$

$$b=2$$

$$\boxed{x=\frac{-3}{5}t+2}$$

x	y
2	3
-1	-1

$$\frac{3+1}{2+1}=\frac{4}{3} \Rightarrow y=\frac{4}{3}x+b$$

$$-1=\frac{4}{3}\cdot(-1)+b$$

$$-1=\frac{-4}{3}+b \Rightarrow b=\frac{1}{3}$$

$$y=\frac{4}{3}x+\frac{1}{3}$$

$$y=\frac{4}{3}\left(-\frac{3}{5}t+2\right)+\frac{1}{3}=-\frac{4}{5}t+\frac{8}{3}+\frac{1}{3}$$

$$\boxed{y=\frac{-4}{5}t+3}$$

original trip: $x = \frac{-3}{5} t + 2$

$y = -\frac{4}{5} t + 3$

Return trip: Replace t with $5-t$.

$x = \frac{-3}{5}(5-t) + 2 = -3 + \frac{3}{5}t + 2 = \frac{3}{5}t - 1$

$y = -\frac{4}{5}(5-t) + 3 = -4 + \frac{4}{5}t + 3 = \frac{4}{5}t - 1.$

Exercise 6:

(a) $x = e^{\sqrt{t}}$, $y = t - \ln(t^2)$ at $t = 1$.

$\frac{dx}{dt} = e^{\sqrt{t}} \cdot \frac{1}{2} t^{-\frac{1}{2}} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}$, $x'(1) = \frac{e^{\sqrt{1}}}{2\sqrt{1}} = \frac{e}{2}$, $x(1) = e$

$\frac{dy}{dt} = 1 - \frac{1}{t^2} \cdot 2t = 1 - \frac{2}{t}$, $y'(1) = 1 - \frac{2}{1} = -1$ $y(1) = 1.$

$\frac{dy}{dx} = \frac{y'(1)}{x'(1)} = \frac{-1}{\frac{e}{2}} = -\frac{2}{e}$ } slope of tangent line.

$(e, 1)$ is a point on the tangent line when $t = 1.$

$$\boxed{y - 1 = \frac{-2}{e}(x - e)}$$

(b) $x = \cos(\theta) + \sin(2\theta)$

$$\frac{dx}{d\theta} = -\sin\theta + \cos(2\theta)\cdot 2 = 2\cos(2\theta) - \sin\theta, \quad x'\left(\frac{\pi}{2}\right) = 2\cos(\pi) - \sin\left(\frac{\pi}{2}\right)$$
$$= -2 - 1 = -3.$$

$y = \cos(\theta)$

$$x\left(\frac{\pi}{2}\right) = 0$$

$$\frac{dy}{d\theta} = -\sin\theta, \quad y'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1, \quad y\left(\frac{\pi}{2}\right) = 0.$$

$$\left.\frac{dy}{dx} = \frac{-1}{-3} = \frac{1}{3}\right\} \Rightarrow$$ slope of tangent line.

$(0, 0)$ is a point on the tangent line when $\theta = \frac{\pi}{2}$.

$$\boxed{y = \frac{1}{3}x}$$

Exercise 7:

(a) $x = e^{\sqrt{t}}, \quad y = t + e^{-t}$

$$\frac{dx}{dt} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}, \quad \frac{dy}{dt} = 1 - \frac{2}{t} \Rightarrow$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{e^{\sqrt{t}}}{2\sqrt{t}}}{1 - \frac{2}{t}} = \frac{\frac{e^{\sqrt{t}}}{2\sqrt{t}}}{\frac{t-2}{t}} = \frac{e^{\sqrt{t}}\cdot t}{2\sqrt{t}\,(t-2)}$$

(b) $x = t^3 - 12t, \quad y = t^2 - 1,$

$$\frac{dx}{dt} = 3t^2 - 12, \quad \frac{dy}{dt} = 2t \Rightarrow \boxed{\frac{dy}{dx} = \frac{3t^2 - 12}{2t}}$$

(c) $x = 4\cos(t)$, $y = \sin(2t)$

$x' = -4\sin(t)$, $y' = 2\cdot\cos(2t) \Rightarrow \frac{dy}{dx} = \frac{2\cos(2t)}{-4\sin(t)} = -\frac{1}{2}\cdot\frac{\cos(2t)}{\sin(t)}$.

Exercise 8: Find $\frac{d^2y}{dx^2}$ for the curve $x = 7 + t^2 + e^t$, $y = \cos(t) + \frac{1}{t}$, $0 < t \leq \pi$.

$$\frac{d^2y}{dx^2} = \frac{x'(t)\,y''(t) - y'(t)\,x''(t)}{(x'(t))^3}$$

$x'(t) = 2t + e^t$ $\qquad y'(t) = -\sin(t) - \frac{1}{t^2}$

$x''(t) = 2 + e^t$ $\qquad y''(t) = -\cos(t) + \frac{2}{t^3}$.

So,
$$\frac{d^2y}{dx^2} = \frac{(2t+e^t)\left(-\cos(t) + \frac{2}{t^3}\right) - \left(-\sin(t) - \frac{1}{t^2}\right)(2+e^t)}{(2t+e^t)^3}$$

Exercise 9:

(a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$

let $u = 1 + t^2$

$du = 2t\,dt$

Arc length is given by $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$.

$$= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\,dt = 6\int_0^1 \sqrt{t^2 + t^4}\,dt = 6\int_0^1 t\sqrt{1+t^2}\,dt =$$

$$= 3\int_1^2 \sqrt{u}\,du = 3\cdot\left\{\frac{2}{3}u^{3/2}\right\}\Big|_1^2 = 2\cdot\left(2^{\frac{3}{2}} - 1\right)$$

(b) $x = 4\cos(t)$, $y = 4\sin(t)$ $\quad 0 \leq t \leq 2\pi$.

By using formula from part a).

Arc length is $\displaystyle\int_0^{2\pi} \sqrt{(-4\sin(t))^2 + (4\cos(t))^2}\, dt = 4 \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t)}\, dt =$

$$= 4 \int_0^{2\pi} \sqrt{1}\, dt = 4 \int_0^{2\pi} dt = 4 \cdot (2\pi - 0) = 8\pi.$$

(c) $x = 3t^2$, $y = 4t^3$, $1 \leq t \leq 3$.

Arc length $= \displaystyle\int_1^3 \sqrt{(6t)^2 + (12t^2)^2}\, dt. = 6 \int_1^3 \sqrt{t^2 + 4t^4}\, dt =$

$= 6 \displaystyle\int_1^3 t\sqrt{1 + 4t^2}\, dt \to$ continue the same way as part a).