Worksheet 21:

Exercise 1: We have the system of particles of masses 4, 2, 5 and 1 located at the coordinates (1,2), (−3,2), (2,−1) and (4,0).

In order to find it we get.

$$M_x = m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4$$

$$= 4 \cdot 2 + 2 \cdot 2 + (-1) \cdot 5 + 1 \cdot 0 = 8 + 4 - 5 = 7.$$

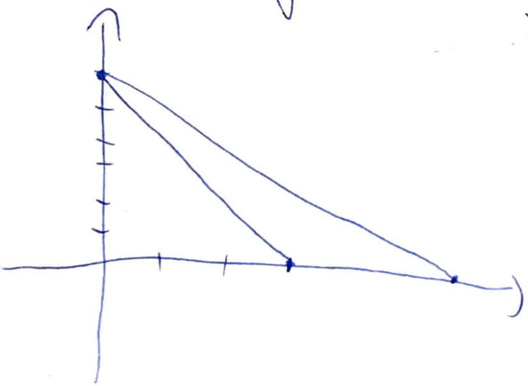
$$M_y = m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 = 4 \cdot 1 + 2 \cdot (-3) + 5 \cdot 2 + 1 \cdot 4 = 4 - 6 + 7 + 4 = 9$$

$$m_{ass} = m_1 + m_2 + m_3 + m_4 = 4 + 2 + 5 + 1 = 12$$

then the center of mass $\left(\frac{M_y}{mass}, \frac{M_x}{mass}\right) = \left(\frac{9}{12}, \frac{7}{12}\right)$

Exercise 2: Point masses of equal size placed at the vertices of the triangle with coordinates (3,0), (b,0) and (0,6) where b > 3

Find center of mass.



$$X_{cm} = \frac{3 + b + 0}{3} = \frac{3+b}{3}$$

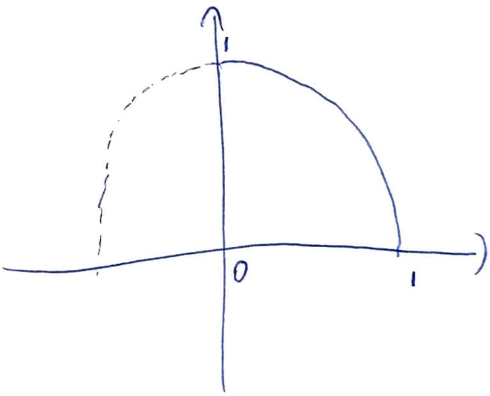
$$Y_{cm} = \frac{0 + 0 + 6}{3} = 2$$

$$COM = \left(\frac{3+b}{3}, 2\right)$$

Exercise 3: Find the centroid of the region under the graph

$y = 1 - x^2$ for $0 \le x \le 1$.

$x = \sqrt{1-y}$ $\boxed{\begin{aligned} u &= 1-y \\ du &= -dy \end{aligned}}$



$$M_y = \rho \int_0^1 x(1-x^2)\,dx$$

$$= \rho \int_0^1 (x - x^3)\,dx$$

$$= \rho \left[\frac{1}{2}x^2 - \frac{x^4}{4} \right] \Big|_0^1$$

$$= \rho \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\rho}{4}$$

$$M_y = \rho \int_0^1 y \sqrt{1-y}\,dy$$

$$= -\rho \int_0^1 (1-u)\,u^{\frac{1}{2}}\,du$$

$$= -\rho \int_0^1 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}} \right) du$$

$$= -\rho \left(\frac{2}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right) \Big|_0^1$$

$$= -\rho \left(\frac{2}{3} - \frac{2}{5} \right)$$

$$M_x = \frac{\rho}{2} \int_0^1 (1 - 2x^2 + x^4)\,dx$$

$$= \frac{\rho}{2} \left(x - \frac{2}{3}x^3 + \frac{x^5}{5} \right) \Big|_0^1 =$$

$$= \frac{\rho}{2} \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{\rho}{2} \cdot \left(\frac{1}{3} + \frac{1}{5} \right)$$

$$= \frac{4\rho}{15}$$

$$M = \rho \int_0^1 (1 - x^2)\,dx$$

$$= \rho \left(x - \frac{1}{3}x^3 \right) \Big|_0^1 =$$

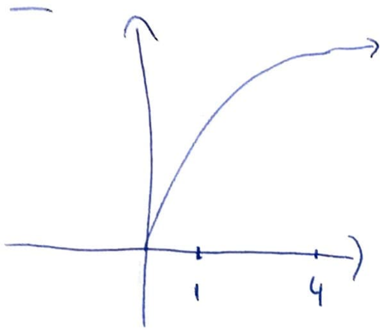
$$= \rho \left(1 - \frac{1}{3} \right) = \frac{2\rho}{3}$$

$$x_{cm} = \frac{\rho/4}{2\rho/3} = \frac{\rho}{4} \cdot \frac{3}{2\rho} = \frac{3}{8}$$

$$y_{cm} = \frac{4\rho}{15} \cdot \frac{3}{2\rho} = \frac{2}{5}$$

$$\boxed{\frac{3}{8},\ \frac{2}{5}}$$

Exercise 4: Centroid under the graph of $f(x)=\sqrt{x}$ for $1 \le x \le 4$.



$$M_y = \rho \int_1^4 x \cdot x^{\frac{1}{2}}\, dx =$$

$$= \rho \int_1^4 x^{\frac{3}{2}}\, dx = \rho \cdot \left(\frac{2}{5} x^{\frac{5}{2}}\right)\Big|_1^4 =$$

$$= \frac{2}{5}\rho(32-1) = \frac{62}{5}\rho$$

$$M_x = \frac{\rho}{2} \int_1^4 x\, dx = \frac{\rho}{2}\left(\frac{1}{2}x^2\right)\Big|_1^4$$

$$= \frac{\rho}{2}\left(8-\frac{1}{2}\right) = \frac{\rho}{2}\cdot\left(\frac{15}{2}\right) = \frac{15\rho}{4}$$

$$M = \rho \int_1^4 \sqrt{x}\, dx = \rho\left(\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_1^4$$

$$= \rho\cdot\left(\frac{16}{3} - \frac{2}{3}\right) = \frac{14\rho}{3}$$

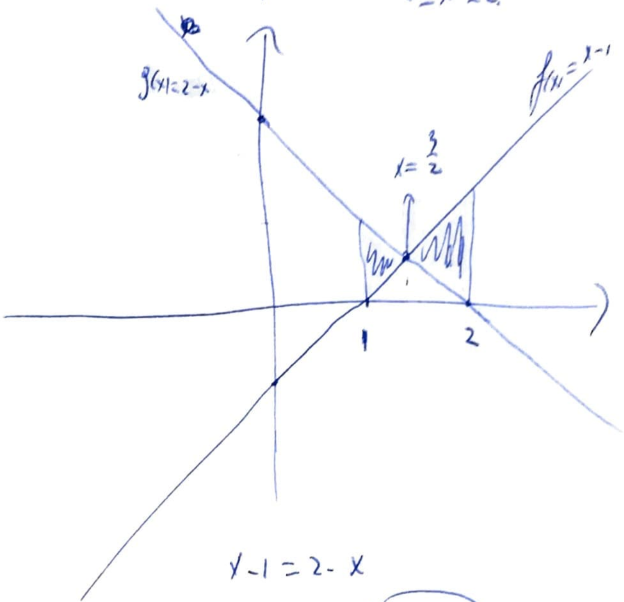
$$x_{cm} = \frac{62\rho}{5} \cdot \frac{3}{14\rho} = \frac{93}{35}$$

$$y_{cm} = \frac{15\rho}{4} \cdot \frac{3}{14\rho} = \frac{45}{56}$$

$$\boxed{\left(\frac{93}{35}, \frac{45}{56}\right)}$$

Exercise 5 : Centroid of the region $f(x) = x-1$ and $g(x) = 2-x$, $1 \le x \le 2$.

$$M_y = P\left(\int_1^{3/2} x\left(2-x-(x-1)\right) dx + \int_{3/2}^{2} x\left(x-1-(2-x)\right) dx \right)$$



$$M_x = \frac{1}{2} P\left(\int_1^{3/2} \left((2-x)^2 - (x-1)^2\right) dx + \int_{3/2}^{2} \left((x-1)^2 - (2-x)^2\right) dx \right)$$

$x-1 = 2-x$

$2x = 3 \Rightarrow \boxed{x = \frac{3}{2}}$

$$M = PA = P\cdot\left(\int_1^{3/2} \left((2-x) - (x-1)\right) dx + \int_{3/2}^{2} \left((x-1) - (2-x)\right) dx \right)$$

$$x_{cm} = \frac{M_y}{M} \qquad y_{cm} = \frac{M_x}{M}$$