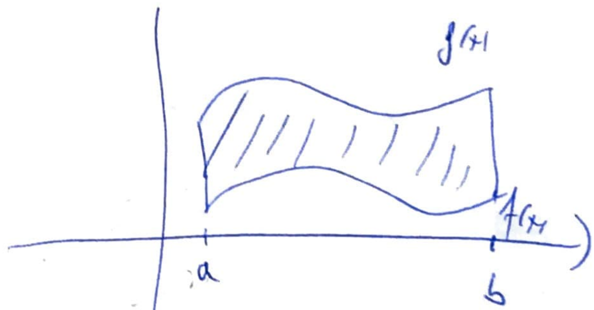
Helpful formulas at Worksheet 21:

(a) Write the formulas for the coordinates of the centroid of a plate with constant density bounded between $x=a$, $x=b$, $f(x)$ and $g(x)$ as in the figure.

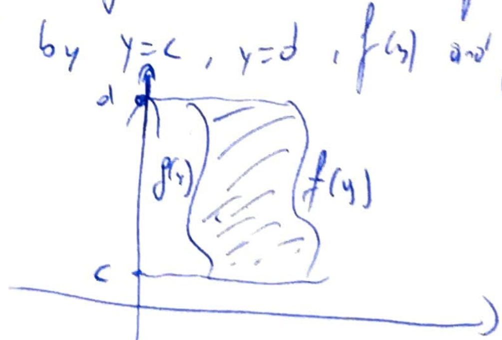


$$M_y = \rho \int_a^b x\left(g(x) - f(x)\right) dx$$

$$M_x = \frac{1}{2}\rho \int_a^b \left(g^2(x) - f^2(x)\right) dx$$

$$x_{cm} = \frac{M_y}{m}, \quad y_{cm} = \frac{M_x}{m} \qquad M = \rho \cdot A = \rho \int_a^b \left(g(x) - f(x)\right) dx.$$

(b) Write the formulas for the coordinates of the centroid of a plate with constant density bounded by $y=c$, $y=d$, $f(y)$ and $g(y)$



$$M_x = \rho \int_c^d y\left(f(y) - g(y)\right)^{\rho} dy$$

$$M_y = \frac{\rho}{2} \int_c^d \left(f^2(y) - g^2(y)\right) dy, \quad x_{cm} = \frac{M_y}{m}$$

$$y_{cm} = \frac{M_x}{m}$$

$$M = \rho \cdot A = \rho \int_c^d \left(f(y) - g(y)\right) dy.$$

Worksheet 20:

Exercise 1:

(a) Assume that $f'(x)$ exists and is continuous on $[a,b]$. Then the arc lengths of $y=f(x)$ over $[a,b]$ is equal to

$$S=\int_a^b \sqrt{1+(f'(x))^2}\,dx.$$

(b) Assume that $f(x)\geq 0$ and that $f'(x)$ exists and is continuous on $[a,b]$. The surface area of the surface obtained by rotating the graph of $f(x)$ about the x-axis for $a\leq x\leq b$ is equal to $S=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$.

if you are rotating about $y=c$ we get

$$S=2\pi\int_a^b (c-f(x))\sqrt{1+(f'(x))^2}\,dx.$$

Exercise 2:

(a) $f(x) = \sin(x)$ from $x=0$ to $x=2$.

$$S = \int_0^2 \sqrt{1+\cos^2(x)}\, dx.$$

(b) $f(x) = x^4$ from $x=2$ to $x=6$.

$$S = \int_2^4 \sqrt{1+(4x^3)^2}\, dx.$$

(c) $x^2+y^2=1 \Leftrightarrow y^2 = 1-x^2 \Leftrightarrow y = \boxed{\sqrt{1-x^2} = f(x)}$

$$f'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot 2x$$

$$= x \cdot (1-x^2)^{-\frac{1}{2}} \Rightarrow S = \int \sqrt{1+\left(x(1-x^2)^{-\frac{1}{2}}\right)^2}\, dx$$

bounds ?

$$\boxed{S = 4\int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\, dx}$$

Exercise 3:

(a) $f(x) = x^{3/2}$ from $x=0$ to $x=2$.

$$S = \int_0^2 \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx = \int_0^2 \sqrt{1+\frac{9}{4}x}\,dx = \int_0^2 \left(1+\frac{9}{4}x\right)^{\frac{1}{2}} dx.$$

$$= \frac{2}{3}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\left(\frac{4}{9}\right)\Big|_0^2 = \frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\Big|_0^2 =$$

$$= \frac{8}{27}\left(1+\frac{9}{4}(2)\right)^{3/2} - \frac{8}{27}\cdot(1)^{3/2} = \frac{8}{27}\cdot\left(\frac{11}{2}\right)^{3/2} - \frac{8}{27} \simeq 3{,}52.$$

(b). $f(x) = \ln(\cos(x))$ from $x=0$ to $x=\frac{\pi}{3}$.

$$f'(x) = \left(\frac{1}{\cos(x)}\right)\cdot(-\sin(x)) \Leftrightarrow (f'(x))^2 = \left(\frac{1}{\cos(x)}\cdot(-\sin(x))\right)^2 = \frac{\sin^2(x)}{\cos^2(x)}.$$

$$S = \int_0^{\frac{\pi}{3}} \sqrt{\frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)}}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{\frac{1}{\cos^2(x)}}\,dx = \int_0^{\frac{\pi}{3}} \sec(x)\,dx =$$

$$= \ln|\sec(x) + \tan(x)|\Big|_0^{\frac{\pi}{3}} = \ln|2+\sqrt{3}| - \cancel{\ln|1|}^{\,0} = \ln|2+\sqrt{3}|.$$

(c) $f(x) = e^x$ from $x = 0$ to $x = 1$.

$$S = \int_0^1 \sqrt{1+(e^x)^2}\,dx = \int_0^1 \sqrt{1+e^{2x}}\,dx.$$

First find $I = \int \sqrt{1+e^{2x}}\,dx$

Substitute $u = -2x \Rightarrow \frac{du}{dx} = -2 \Rightarrow \boxed{dx = -\frac{1}{2}\,du}$

$I = -\frac{1}{2}\int \sqrt{e^{-u}+1}\,du \Rightarrow$ substitute $t = e^{-u}+1 \Rightarrow \frac{dt}{du} = -e^{-u} \Rightarrow$

$du = -e^{u}\,dt \Rightarrow$

$-\int \frac{\sqrt{t}}{t-1}\,dt \Rightarrow$ substitute $w = \sqrt{t} \Rightarrow \frac{dw}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow$

$dt = 2\sqrt{t}\,dw$ (use $t = w^2$)

$$= 2\int \frac{w^2}{w^2-1}\,dw = 2\int\left(\frac{w^2-1}{w^2-1} + \frac{1}{w^2-1}\right)dw = 2\int\left(1+\frac{1}{w^2-1}\right)dw$$

other steps as an exercise $\Rightarrow I = \frac{\ln\left(\sqrt{e^{2x}+1}-1\right) - \ln\left(\sqrt{e^{2x}+1}+1\right)}{2} + \sqrt{e^{2x}+1} +$

$$S = \int_0^1 \sqrt{1+e^{2x}}\,dx = \left(\frac{\ln\left(\sqrt{e^{2x}+1}-1\right) - \ln\left(\sqrt{e^{2x}+1}+1\right)}{2} + \sqrt{e^{2x}+1}\right)\Bigg|_0^1$$

Exercise 4:

$f(x) = 2x+1$ from $x=0$ to $x=t$ we need to find $S(4)$.

$f'(x)=2$ $\Rightarrow$ $S(t) = \int_0^t \sqrt{1+4}\, dx = \int_0^t \sqrt{5}\, dt$ $\Rightarrow$

$(f'(x))^2 = 4$

$$S(4) = \int_0^4 \sqrt{5}\, dt = \sqrt{5}\cdot x \Big|_0^4 = \sqrt{5}\cdot 4.$$

Exercise 5:

(a) $y = x$, $[0,4]$.

$$S = 2\pi \int_0^4 x\sqrt{1+1}\, dx = 2\pi \int_0^4 \sqrt{2}\, x\, dx = 2\pi \cdot \frac{\sqrt{2}}{2} x^2 \Big|_0^4 = \sqrt{2}\pi \cdot x^2 \Big|_0^4$$

$$= \sqrt{2}\pi \cdot (16) = 16\sqrt{2}\pi.$$

(b) $y = x^3$, $[0,2]$.

$y' = 3x^2$ $\Rightarrow$ $S = 2\pi \int_0^2 x^3 \cdot \sqrt{1+9x^4}\, dx$

$(y')^2 = (3x^2)^2$

$= 9x^4$

first we need $I = \int x^3 \sqrt{1+9x^4}\, dx$

let $u = 1+9x^4 \Rightarrow du = (1+9x^4)'\, dx = 36x^3\, dx \Rightarrow \boxed{\frac{du}{36} = x^3\, dx} \Rightarrow$

$$I = \int \sqrt{u} \cdot \frac{du}{36} = \frac{1}{36}\int u^{\frac{1}{2}}\, du = \frac{1}{36}\cdot \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{36}\cdot u^{3/2}\cdot \frac{2}{3} + c \Rightarrow$$

$$I = \frac{2}{36\cdot 3}\cdot (1+9x^4)^{3/2} + c \Rightarrow$$

$$S = 2\pi \cdot \frac{2}{36\cdot 3}\cdot (1+9x^4)^{3/2} \Big|_0^2$$

(c) $y = \left(4 - x^{2/3}\right)^{3/2}$, $[0,8]$.

$y' = \frac{3}{2}\left(4 - x^{2/3}\right)^{\frac{1}{2}} \cdot \left(-\frac{2}{3} x^{-\frac{1}{3}}\right) = -x^{-1/3}\left(4 - x^{2/3}\right)^{\frac{1}{2}}$.

$y'^2 = x^{-\frac{2}{3}}\left(4 - x^{\frac{2}{3}}\right) = \boxed{4x^{-\frac{2}{3}} - 1 = y'^2} \Rightarrow$

$S = 2\pi \int_0^8 \left(4 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \cdot \sqrt{4x^{-\frac{2}{3}}}\, dx =$

$= 2\pi \int_0^8 \left(4 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \cdot 2x^{-\frac{1}{3}}\, dx =$

let $u = 4 - x^{2/3}$, $\boxed{\begin{array}{l} u(8)=0 \\ u(0)=4 \end{array}}$

$du = -\frac{2}{3} x^{-\frac{1}{3}}\, dx$

$-\frac{3}{2}\, du = x^{-\frac{1}{3}}\, dx$

$= -6\pi \int_4^0 u^{\frac{3}{2}}\, du =$

$= -6\pi \left(\frac{2}{5} \cdot u^{\frac{5}{2}}\right)\Big|_4^0 =$

$= -6\pi(0) - (-6\pi)\left(\frac{2}{5} \cdot 32\right)$

$= 6\pi\left(\frac{64}{5}\right) \simeq 241{,}2$

(d) $y = e^{-x}$, $[0,1]$

$y' = -e^{-x} \Rightarrow y'^2 = (-e^{-x})^2 = e^{-2x}$

let $u = e^{-x} \Rightarrow$

$S = 2\pi \int_0^1 e^{-x} \sqrt{1 + e^{-2x}}\, dx.$

$du = -e^{-x}\, dx \Rightarrow$

$-du = e^{-x}\, dx.$

$\boxed{\begin{array}{l} e^{-0} = 1 \\ e^{-1} = \frac{1}{e} \end{array}}$

$S = 2\pi \int_1^{\frac{1}{e}} -\sqrt{1+u^2}\, du$

Then do the substitution $u = tg(t)$ and continue.

(e) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$, $[1,e]$.

$y' = \frac{1}{4} \cdot 2x - \frac{1}{2}\frac{1}{x} = \frac{1}{2}x - \frac{1}{2x}$

$\Rightarrow y'^2 = \left(\frac{1}{2}x - \frac{1}{2x}\right)^2$

$S = 2\pi \int_1^e \left(\frac{1}{4}x^2 - \frac{1}{2}\ln(x)\right)\sqrt{1 + \left(\frac{1}{2}x - \frac{1}{2x}\right)^2}\, dx$

First find $I = \int \left(\frac{x^2}{4} - \frac{\ln(x)}{2}\right)\sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, dx. =$

$= -\frac{1}{8} \int \sqrt{\left(x - \frac{1}{x}\right)^2 + 4} \cdot \left(2\ln(x) - x^2\right) dx$

Now we can see that $\sqrt{\left(x-\frac{1}{x}\right)^2+4}=\sqrt{\left(\frac{x^2-1}{x}\right)^2+4}=$

$$=\sqrt{\frac{x^4-2x^2+1}{x^2}+4}=\sqrt{\frac{x^4-2x^2+1+4x^2}{x^2}}=\sqrt{\frac{x^4+2x^2+1}{x^2}}=$$

$$=\sqrt{\frac{(x^2+1)^2}{x^2}}=\frac{x^2+1}{x}\implies$$

$$\int\sqrt{\left(x-\frac{1}{x}\right)^2+4}\,\left(2\ln(x)-x^2\right)dx=\int\frac{(x^2+1)(2\ln(x)-x^2)}{x}\,dx=$$

$$=\int\frac{2x^2\ln(x)+2\ln(x)-x^4-x^2}{x}\,dx=\int\left(2x\ln(x)+\frac{2\ln(x)}{x}-x^3-x\right)dx=$$

$$=2\int x\ln(x)\,dx+2\int\frac{\ln(x)}{x}\,dx-\int x^3dx-\int x\,dx=$$

$$=2\left(\frac{x^2\ln(x)}{2}-\int\frac{x}{2}\,dx\right)+2\int\ln(x)\,d(\ln(x))-\frac{x^4}{4}-\frac{x^2}{2}+C=$$

$$=2\left(\frac{x^2\ln(x)}{2}-\frac{1}{2}\cdot\frac{x^2}{2}\right)+2\cdot\frac{(\ln(x))^2}{2}-\frac{x^4}{4}-\frac{x^2}{2}+C=$$

$$=x^2\ln(x)-\frac{x^2}{2}+(\ln(x))^2-\frac{x^4}{4}-\frac{x^2}{2}+C$$

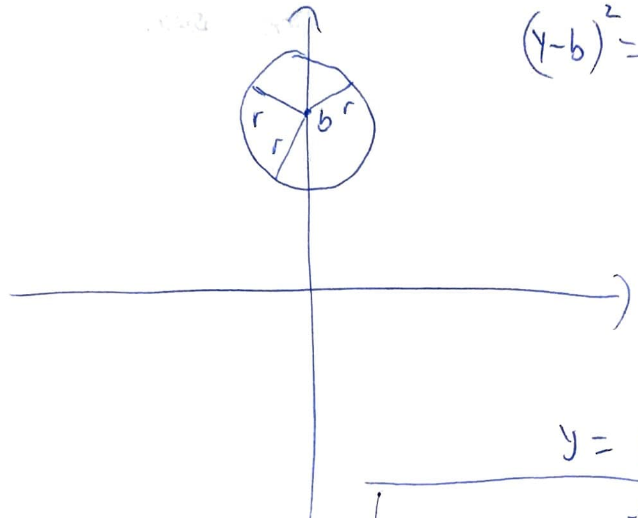
After substituting at the beginning and being careful with the other constants from the first steps we get:

$$I = \frac{-4\ln^2(x) - 4x^2\ln(x) + x^4 + 4x^2}{32} + C.$$

$$\Rightarrow S = 2\pi \cdot \left(\frac{-4\ln^2(x) - 4x^2\ln(x) + x^4 + x^2}{32} \right) \Bigg|_1^e .$$

(g) Surface area of the torus about the x-axis.

$x^2 + (y-b)^2 = r^2$. $\Rightarrow$



$(y-b)^2 = r^2 - x^2 \Rightarrow y - b = \pm\sqrt{r^2-x^2}$

$y = b \pm \sqrt{r^2-x^2}$

outer surface is generated by

$y = b + \sqrt{r^2-x^2}$

$$\boxed{y' = \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{r^2-x^2}}} \Rightarrow$$

for outer surface $\Rightarrow$ $S = 2\pi \int_{-r}^{r} (b + \sqrt{r^2-x^2}) \cdot \sqrt{1 + \frac{x^2}{(r^2-x^2)}}\, dx$

By using symmetry $S_{out} = 2S = 2 \cdot 2\pi \int_{-r}^{r} (b + \sqrt{r^2-x^2}) \sqrt{1 + \frac{x^2}{r^2-x^2}}\, dx$

$S_{out} = 4\pi \int_{-r}^{r} (b + \sqrt{r^2-x^2}) \sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\, dx = 4\pi \int_{-r}^{r} (b + \sqrt{r^2-x^2}) \frac{x}{\sqrt{r^2-x^2}}\, dx.$

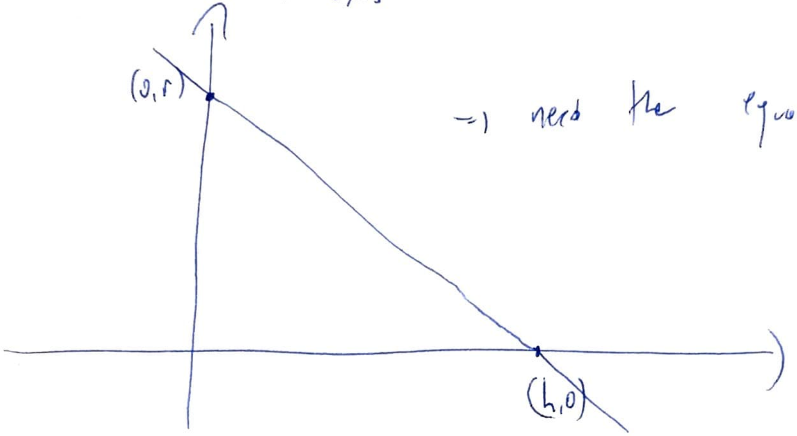
Then substitute $x = r\sin(t)$ and then calculate it.

Same with $S_{in} \Rightarrow y = b - \sqrt{r^2-x^2}$ then

$$\boxed{S = S_{out} + S_{in}}$$

(h) the surface area given by rotating the graph of $f(x)$ around the x-axis

↓ right cone



$\Rightarrow$ need the equation for the line,

$y = mx + b$

$r = m \cdot 0 + b \Rightarrow r = b$

$0 = m \cdot h + b \Rightarrow$

$mh = -b = -r \Rightarrow m = \frac{-r}{h}$

$\boxed{y = \frac{-r}{h} x + r}$ $\quad y' = \frac{-r}{h} \Rightarrow y'^2 = \frac{r^2}{h^2} \Rightarrow$

$$S = 2\pi \int_0^h \left(\frac{-r}{h} \cdot x + r\right) \cdot \sqrt{1 + \frac{r^2}{h^2}}\, dx = 2\pi \sqrt{1 + \frac{r^2}{h^2}} \cdot \int_0^h \left(-\frac{r}{h} x + r\right) dx$$

$$= 2\pi r \sqrt{1 + \frac{r^2}{h^2}} \int_0^h \left(1 - \frac{x}{h}\right) dx = 2\pi r \cdot \sqrt{1 + \frac{r^2}{h^2}} \left(x - \frac{x^2}{2h}\right)\Big|_0^h =$$

$$= 2\pi r \cdot \sqrt{1 + \frac{r^2}{h^2}} \cdot \left(h - \frac{h^2}{2h}\right) = 2\pi r \sqrt{1 + \frac{r^2}{h^2}} \cdot \left(h - \frac{h}{2}\right) = 2\pi r \sqrt{1 + \frac{r^2}{h^2}} \cdot \frac{1}{2} h$$

$$= \pi r \cdot \sqrt{h^2\left(1 + \frac{r^2}{h^2}\right)} = \boxed{\pi r \cdot \sqrt{h^2 + r^2}}$$