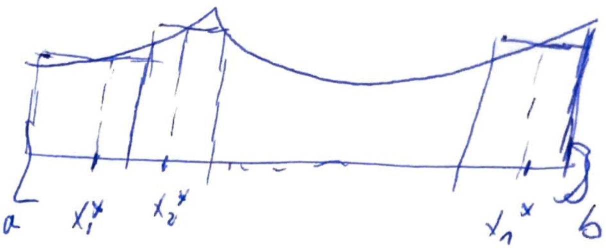
Worksheet 17:

Exercise 1: $f(x)$ on $[a,b]$.

$$f_{avg} = \frac{\int_a^b f(x)\,dx}{b-a}$$



How to get it? We usually have that

$$Avg_n = \frac{f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)}{n}$$

if all subintervals are of size $\Delta x = \frac{b-a}{n}$, then we get

$n = \frac{b-a}{\Delta x}$ and so $Avg_n = \frac{\sum_{i=1}^{n} f(x_i^*)}{\frac{b-a}{\Delta x}} = \sum_{i=1}^{n} f(x_i^*) \cdot \frac{\Delta x}{b-a} =$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \Rightarrow$$

$$f_{avg} = \lim_{n\to\infty} Avg_n = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Exercise 2 :

(a) $f(x) = x^3$, $[0,4]$.

$$\frac{1}{4-0}\int_0^4 x^3\,dx = \frac{1}{4}\cdot\frac{x^4}{4}\Big|_0^4 = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{4^4}{1} - \frac{1}{4}\cdot\frac{1}{4}\cdot 0^4$$

$$= \frac{1}{4^2}\cdot 4^4 = 4^2 = 16.$$

(b) $f(x) = x^3$ $[-1,1]$

$$\frac{1}{1-(-1)}\int_{-1}^{1} x^3\,dx = \frac{1}{2}\cdot\frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{2}\left(\overbrace{\frac{1}{4}-\frac{(-1)^4}{4}}^{=0}\right) = 0.$$

(c) $f(x) = \cos(x)$. $\left[0, \frac{\pi}{6}\right]$.

$$f_{avg} = \frac{1}{\frac{\pi}{6}-0}\int_0^{\pi/6}\cos(x)\,dx = \frac{1}{\frac{\pi}{6}}\cdot\sin(x)\Big|_0^{\pi/6} = \frac{6}{\pi}\left(\sin\left(\frac{\pi}{6}\right) - \overset{=0}{\sin(0)}\right)$$

$$= \frac{6}{\pi}\cdot\frac{1}{2} = \frac{3}{\pi}.$$

(d) $f(x) = \frac{1}{x^2+1}$ $[-1, 1]$.

$$f_{avg} = \frac{1}{1-(-1)} \int_{-1}^{1} \frac{1}{x^2+1}\, dx = \frac{1}{2} \operatorname{arctg}(x)\Big|_{-1}^{1} =$$

$$= \frac{1}{2}\left(\operatorname{arctg}(1) - \operatorname{arctg}(-1)\right) = \frac{1}{2}\cdot\left(\frac{\pi}{4} - \frac{3\pi}{4}\right) = \frac{1}{2}\cdot\frac{-2\pi}{4} = \frac{-\pi}{4}.$$

(e) $f(x) = \frac{\sin(\pi/x)}{x^2}$, $[1, 2]$.

$$d\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)' \cdot dx = \frac{-1}{x^2}\, dx$$

$$f_{avg} = \frac{1}{2-1} \int_{1}^{2} \frac{\sin\frac{\pi}{x}}{x^2}\, dx = 1\cdot \int_{1}^{2} (-1) \sin\left(\frac{\pi}{x}\right)\cdot d\left(\frac{1}{x}\right) = -\int_{1}^{2} \sin\left(\frac{\pi}{x}\right) d\left(\frac{1}{x}\right)$$

$$= \frac{-1}{\pi} \int_{1}^{2} \sin\left(\frac{\pi}{x}\right) d\left(\frac{\pi}{x}\right) = -\frac{1}{\pi}\left(-\cos\left(\frac{\pi}{x}\right)\Big|_{1}^{2}\right) =$$

$$= -\frac{1}{\pi}\left(-\cos\left(\frac{\pi}{2}\right) + \cos(\pi)\right) = -\frac{1}{\pi}\cdot(0-1) = \frac{1}{\pi}.$$

(f) $f(x) = e^{-nx}$, $[-1, 1]$.

$$f_{avg} = \frac{1}{1-(-1)} \int_{-1}^{1} e^{-nx}\, dx = \frac{1}{2}\cdot\frac{-1}{n} \int_{-1}^{1} e^{-nx}\, d(-nx) = \frac{-1}{2n} e^{-nx}\Big|_{-1}^{1} =$$

$$= \frac{-1}{2n}\left(e^{-n} - e^{-n(-1)}\right) = \frac{1}{2n}\left(e^{n} - e^{-n}\right)$$

(g) $f(x) = 2x^3 - 6x^2$, $[-1, 3]$.

$$f_{avg} = \frac{1}{3-(-1)} \int_{-1}^{3} (2x^3 - 6x^2)\, dx = \frac{1}{4} \left(2\frac{x^4}{4} - 6\frac{x^3}{3} \right) \Big|_{-1}^{3}$$

$$= \frac{1}{4} \left(\frac{x^4}{2} - 2x^3 \right) \Big|_{-1}^{3} = \frac{1}{4} \left(\left(\frac{3^4}{2} - 2 \cdot 3^3 \right) - \left(\frac{1}{2} + 2 \right) \right)$$

$$= \frac{1}{4} \left(\frac{3^4 - 4 \cdot 3^3}{2} - \frac{1}{2} + \frac{4}{2} \right) =$$

$$= \frac{1}{4} \left(\frac{3^3(3-4) - 1 + 4}{2} \right) = \frac{1}{4} \cdot \frac{27 \cdot (-1) - 1 + 4}{2} =$$

$$= \frac{-27 - 1 + 4}{8} = \frac{-24}{8}$$

---

h) $f(x) = x^n$, for $n \geq 0$ $[0, 1]$

$$f_{avg} = \frac{1}{1-0} \int_0^1 x^n\, dx = \frac{1}{1} \cdot \frac{x^{n+1}}{n+1} \Big|_0^1 = 1 \cdot \frac{1}{n+1} = \frac{1}{n+1}$$

Exercise 3:

After 9.am we have $T(t) = 50 + 14 \sin\left(\frac{\pi t}{12}\right)$

Then average temperature is Note $9.p.m = 21\ a.m.$

$$T_{avg} = \frac{1}{21-9} \int_9^{21} \left(50 + 14 \sin \frac{\pi t}{12}\right) dt =$$

$$= \frac{1}{12}\left(50t\Big|_9^{21} + 14 \cdot \frac{12}{\pi} \int_9^{21} \sin \frac{\pi t}{12}\, d\left(\frac{\pi t}{12}\right)\right) =$$

$$= \frac{1}{12}\left(50(21-9) + \frac{14 \cdot 12}{\pi} \cdot \left(-\cos \frac{\pi t}{12}\right)\Big|_9^{21}\right) =$$

$$= \frac{1}{12}\left(50 \cdot 12 - \frac{14 \cdot 12}{\pi} \cdot \left(\cos \frac{9\pi}{12} - \cos \frac{21\pi}{12}\right)\right)$$

Exercise 4:

$$V(r) = \frac{P}{4\eta l}\left(R^2 - r^2\right)$$

$$V_{avg} = \frac{1}{R-0}\int_0^R \frac{P}{4\eta l}\left(R^2 - r^2\right) dr = \frac{1}{R}\cdot\frac{P}{4\eta l}\int_0^R \left(R^2 - r^2\right) dr =$$

$$= \frac{P}{R\cdot 4\eta l}\left(R^2\cdot r - \frac{r^3}{3}\right)\Bigg|_0^R = \frac{P}{R\cdot 4\eta l}\left(R^3 - \frac{R^3}{3}\right) =$$

$$= \frac{P}{R4\eta l}\left(\frac{2R^3}{3}\right) = \boxed{\frac{2PR^3}{12\cdot\eta l} = V_{avg}}$$

To find the maximum velocity we first need to find the derivative and so.

$$v'(r) = \left(\frac{P}{4\eta l}R^2 - \frac{P}{4\eta l}r^2\right)' = -2\cdot\frac{P}{4\eta l}r = 0 \quad \text{and so}$$

$$\boxed{r=0} \Rightarrow V_{max} = V(0) = \frac{P}{4\eta l}\left(R^2 - 0^2\right) = \boxed{\frac{PR^2}{4\eta l} = V_{max}}$$

Exercise 5 :

$$f(t) = \frac{1}{2} \sin\left(\frac{2\pi t}{5}\right)$$

$$f_{avg} = \frac{1}{5-0} \int_0^5 \frac{1}{2} \sin\left(\frac{2\pi t}{5}\right) dt = \frac{1}{5} \int_0^5 \frac{1}{2} \sin\left(\frac{2\pi t}{5}\right) dt =$$

$$= \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{5}{2\pi} \int_0^5 \sin\left(\frac{2\pi t}{5}\right) d\left(\frac{2\pi t}{5}\right) =$$

$$= \frac{1}{10} \cdot \frac{5}{2\pi} \cdot \left(-\cos\left(\frac{2\pi t}{5}\right) \Big|_0^5 \right) =$$

$$= \frac{5}{20\pi} \cdot \left(1 - \cos \frac{2\pi \cdot 5}{5} \right) = \frac{5}{20\pi} \left(1 - \cos 2\pi \right)$$

$$= 0.$$