Worksheet 16:

Exercise 1:

(a) $a_n = \frac{(-1)^n \cdot n}{n! + 1}$

$a_0 = \frac{(-1)^0 \cdot 0}{0! + 1} = 0$, $a_1 = \frac{(-1)^1 \cdot 1}{1! + 1} = \frac{-1}{2}$,

$a_2 = \frac{(-1)^2 \cdot 2}{2! + 1} = \frac{2}{3}$, $a_3 = \frac{(-1)^3 \cdot 3}{3! + 1} = \frac{-3}{7}$.

$a_4 = \frac{(-1)^4 \cdot 4}{4! + 1} = \frac{4}{25}$, $a_5 = \frac{(-1)^5 \cdot 5}{5! + 1} = \frac{-5}{121}$

(b) $a_1 = 6$, $a_{n+1} = \frac{a_n}{n}$.

$a_2 = \frac{a_1}{1} = \frac{6}{1} = 6$, $a_3 = \frac{a_2}{2} = \frac{6}{2} = 3$, $a_4 = \frac{a_3}{3} = \frac{3}{3} = 1$

$a_5 = \frac{a_4}{4} = \frac{1}{4}$

Exercise 2:

(a) $a_n = 3^n \cdot 7^{-n}$, since $\lim_{n\to\infty} 3^n \cdot 7^{-n} = \lim_{n\to\infty} \left(\frac{3}{7}\right)^n = 0$ it converges.

(b) $a_n = \frac{(-1)^{n+1} \cdot n}{n + \sqrt{n}}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1} \cdot n}{n + \sqrt{n}} = \lim_{n\to\infty} \frac{(-1)^{n+1} \cdot \cancel{n}}{\cancel{n}(1 + \underbrace{\frac{\sqrt{n}}{n}}_{\xrightarrow{n\to\infty} 0})} =$ DNE because it is either -1

$\Rightarrow$ divergent.

(c) $a_n = \dfrac{\ln(n)}{\ln(2n)}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{\ln(n)}{\ln(2n)} \overset{\text{L'hopital}}{=} \lim_{n\to\infty} \dfrac{\frac{1}{n}}{\frac{1}{2n}\cdot 2} = 1$

$\Downarrow$

convergent

(d) $a_n = \dfrac{\cos^2(n)}{2^n}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{\cos^2(n)}{2^n} = 0 \Rightarrow$ convergent

Exercise 3:

$\sum_{n=1}^{\infty} a_n = 2$ means that the series is convergent and converges to 2. So if we take $S_n = \sum_{k=1}^{n} a_k$ then $\lim_{n\to\infty} S_n = 2$.

Exercise 4:

(a) $\sum_{n=1}^{\infty} \dfrac{(-4)^{n-1}}{3^n} = \dfrac{1}{3} \sum_{n=1}^{\infty} \left(\dfrac{-4}{3}\right)^{n-1} = \dfrac{1}{3} \sum_{n=0}^{\infty} \left(\dfrac{-4}{3}\right)^n$

Since $\lim_{n\to\infty} \left(\dfrac{-4}{3}\right)^n \neq 0$ then By Divergence Test it is divergent.

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(b) $\sum_{n=1}^{\infty} \dfrac{6\cdot 2^{n-1}}{3^n} = \sum_{n=1}^{\infty} \dfrac{3\cdot 2\cdot 2^{n-1}}{3^n} = 3 \sum_{n=1}^{\infty} \left(\dfrac{2}{3}\right)^n \to$ convergent since Geometric Series with $\dfrac{2}{3} < 1$.

Exercise 5:

(a) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ Since we know that $\frac{d}{dx}\ln x = \frac{1}{x}$ then

$$\frac{d}{dx}\left(\ln(\ln(x))\right) = \frac{1}{\ln(x)} \cdot \frac{d}{dx}\ln(x) = \frac{1}{\ln(x)} \cdot \frac{1}{x}$$. Thus

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{A\to\infty} \int_2^A \frac{1}{x\ln(x)}\,dx = \lim_{A\to\infty} \ln(\ln(x))\Big|_2^A = \infty.$$

Thus it diverges.

(b) $\sum_{n=1}^{\infty} \underbrace{\frac{7\sqrt{n}}{5n^{3/2}+3n-2}}_{a_n}$ $\quad b_n = \frac{\sqrt{n}}{n^{3/2}} = \frac{n^{1/2}}{n^{3/2}} = \frac{1}{n} \to \sum_{n=1}^{\infty} \frac{1}{n}$ divergent

By limit test:

$$\lim_{n\to\infty} \frac{7\sqrt{n}}{5n^{3/2}+3n-2} \cdot \frac{n^{3/2}}{\sqrt{n}} = \frac{7}{5} < \infty$$ Thus $\sum_{n=1}^{\infty} \frac{7\sqrt{n}}{5n^{3/2}+3n-2}$ diverges

(c) $\sum_{n=1}^{\infty} n!\,e^{-8n}$, Use Ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{e^{8n+8}} \cdot \frac{e^{8n}}{n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!} \cdot \frac{\cancel{e^{8n}}}{\cancel{e^{8n}} \cdot e^8}\right|$$

$$= \lim_{n\to\infty}\left|n \cdot \frac{1}{e^8}\right| = \infty$$ $\Rightarrow$ divergent.

(d) $\sum_{n=1}^{\infty} \left(\frac{\ln(n)}{5n+7}\right)^n$

Use Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln(n)}{5n+7}\right)^n} = \lim_{n\to\infty} \frac{\ln(n)}{5n+7} = 0 < 1$$

Since $L < 1$ then it is convergent.

(e) $\sum_{n=1}^{\infty} \frac{9^n}{9n}$, Since $\lim_{n\to\infty} \frac{9^n}{9n} = \infty \neq 0$ then By Divergence test it diverges.

(f) $\sum_{n=1}^{\infty} (-1)^{n+1} n \cdot e^{-n}$

- Alternating.
- $\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} = 0$

By AST it is convergent.

(g) $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan(n)$,

- Alternating.
- $\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0$ then By AST it diverges.

Exercise 6 :

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1} = \sum_{n=1}^{\infty} (-1)^n a_n$

$\left.\begin{array}{l} \bullet) \text{ Alternating} \\ \bullet) \lim_{n\to\infty} \frac{1}{5n+1} = 0 \end{array}\right\} \Rightarrow$ By AST it is convergent.

Now let's check $\sum_{n=1}^{\infty} |(-1)^n \cdot a_n| = \sum_{n=1}^{\infty} \frac{1}{5n+1}$ $\Rightarrow$ By limit test with $b_n = \frac{1}{5n}$ $\Leftarrow$ divergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{5n+1}$ is divergent.

Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$ converges conditionally.

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b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

$\left.\begin{array}{l} \bullet) \text{ Alternating} \\ \bullet) \lim_{n\to\infty} \frac{1}{n^3+1} = 0 \end{array}\right\} \Rightarrow$ By AST it is convergent.

Check. $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3+1}$ $\Rightarrow$ Since it is convergent

then it converges absolutely.

(c) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$

•) Alternating

•) $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1 \neq 0$

it is divergent.

(d) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ By Root test

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{n^{4n}}} = \lim_{n\to\infty} \frac{n!}{n^4} = \infty > 1$

Thus it diverges.

Exercise 7:

(a) $\sum_{n=1}^{\infty} \frac{x^n}{4^n \cdot n^4}$ Use Ratio test

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{4^{n+1}(n+1)^4} \cdot \frac{4^n \cdot n^4}{x^n} \right| =$

$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{4^n}{4^{n+1}} \cdot \underbrace{\frac{n^4}{(n+1)^4}}_{\xrightarrow{n\to\infty} 1} \right| = \left| \frac{x}{4} \right| < 1$

$-1 < \frac{x}{4} < 1 \iff -4 < x < 4 \Rightarrow R = \frac{4-(-4)}{2}$

$\boxed{R = \frac{8}{2} = 4}$

check $x=-4$ and $x=4$.

$x=-4$; $\sum_{n=1}^{\infty} \frac{(-4)^n}{4^n \cdot n^4} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4^n}{4^n \cdot n^4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \Rightarrow$ Convergent

$x=4$; $\sum_{n=1}^{\infty} \frac{4^n}{4^n \cdot n^4} = \sum_{n=1}^{\infty} \frac{1}{n^4} \Rightarrow$ convergent.

Thus Interval of convergence.

$-4 \le x \le 4 \Leftrightarrow [-4, 4]$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot x^n}{n^2}$

Use Ratio test;

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \underbrace{\frac{n^2}{(n+1)^2}}_{\xrightarrow{n\to\infty} 1} \right|$$

$= |x| < 1.$

$-1 < x < 1 \Rightarrow R = 1.$

$x=-1$; $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow$ convergent;

$x=1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ convergent $\Rightarrow [-1, 1]$.

(c) $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$ Use Ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(5x-4)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(5x-4)^n} \right|$$

$$= |5x-4| < 1 \Rightarrow$$

$$-1 < 5x-4 < 1$$

$$3 < 5x < 5$$

$$\boxed{\frac{3}{5} < x < \frac{5}{5} = 1}$$

$$R = \frac{1-\frac{3}{5}}{2} = \frac{\frac{2}{5}}{2} = \boxed{\frac{1}{5} = R}$$

check:

$x = \frac{3}{5}$: $\sum_{n=1}^{\infty} \frac{(3-4)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \Rightarrow$ convergent

$x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(5-4)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3} \Rightarrow$ convergent.

$$\left[\frac{3}{5}, 1\right].$$

Exercise 8 i

Use $\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n$

(a) $f(x) = \frac{5}{1-4x^2} =$

$$= 5 \cdot \left(1 + 4x^2 + (4x^2)^2 + \dots\right) = 5 \sum_{n=0}^{\infty} (4x^2)^2 = \sum_{n=0}^{\infty} 5 \cdot 16 \cdot x^2$$

$$= \sum_{n=0}^{\infty} 80 x^2$$

(b). $f(x) = \frac{x^2}{x^4+16} = \frac{x^2}{16\left(1 + \frac{x^4}{16}\right)} = \frac{x^2}{16 \cdot \left(1 + \left(\frac{x^2}{4}\right)^2\right)} = \frac{\frac{x^2}{16}}{1 + \left(\frac{x^2}{4}\right)^2}$

$$= \frac{x^2}{16} \cdot \sum_{n=0}^{\infty} \left(\frac{-x^2}{4}\right)^n = \sum_{n=1}^{\infty} \frac{x^2}{\underset{4^2}{16}} \cdot \frac{(-1)^n \cdot x^{2n}}{4^n} =$$

$$\boxed{= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot x^{2n+2}}{4^{n+2}}}$$

(c) $f(x) = \frac{3}{2+2x} = \frac{3}{2(1+x)} = \frac{\frac{3}{2}}{1-(-x)} =$

$$= \frac{3}{2} \sum_{n=0}^{\infty} (-x)^n = \sum_{n=1}^{\infty} \frac{3}{2} \cdot (-1)^n \cdot x^n.$$

(d) $f(x) = e^{-x^2}$, $e^x = \sum_{n=0} \frac{x^n}{n!}$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}$$

Exercise 9: $\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \, dt =$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1} \Rightarrow \text{Radius of convergence } [-1, 1].$$

Exercise 10:

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{1}{4!} \cdot x^4 + \cdots$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2}$$

$$\lim_{x \to 0} \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{1}{4!} x^4 + \cdots\right)}{x^2} = \lim_{x \to 0} \frac{\cancel{x^2}\left(\frac{1}{2} + \frac{1}{4!} x^2 + \cdots\right)}{\cancel{x^2}}$$

$$= \frac{1}{2}.$$

Extra i

Exercise 1: Determine the limit of the sequence or state it is divergent.

(a) $a_n = \sqrt{n+3} - \sqrt{n}$ $\Rightarrow$ conjugate. $\dfrac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$

$$= \frac{n+3-n}{\sqrt{n+3}+\sqrt{n}} = \frac{3}{\sqrt{n+3}+\sqrt{n}}$$

$\lim_{n\to\infty} a_n = 0$

$\Rightarrow$ converges.

(b) $a_n = \dfrac{\cos n}{n}$ Since $-1 \le \cos n \le 1$

then $\lim_{n\to\infty} \dfrac{\cos n}{n} = 0.$ $\Rightarrow$ converges

(c) $a_n = \dfrac{e^n + (-3)^n}{5^n}$ $\lim_{n\to\infty} \dfrac{e^n}{5^n} + \lim_{n\to\infty} \dfrac{(-3)^n}{5^n} = 0 + 0 = 0.$ converges

(d) $a_n = n^{1/n}$ $\lim_{n\to\infty} e^{\frac{\ln(n)}{n}}$; $\lim_{n\to\infty} \dfrac{\ln(n)}{n} \overset{\text{L'hopital}}{=} \lim_{n\to\infty} \dfrac{1}{n} = 0.$

$\Rightarrow \lim_{n\to\infty} e^{\frac{\ln(n)}{n}} = e^0 = 1$ $\Rightarrow$ convergent.

Exercise 2 :

Determine whether or not $\sum_{n=2}^{\infty}\left(1-\sqrt{1-\frac{1}{n^2}}\right)$ converges

conjugate :

$$\sum_{n=2}^{\infty}\frac{1-\left(1-\frac{1}{n^2}\right)}{1+\sqrt{1-\frac{1}{n^2}}}=\sum_{n=2}^{\infty}\frac{1}{n^2+n^2\sqrt{1-\frac{1}{n^2}}}$$

converges by the p-test.

Exercise 3 :

Evaluate : $\sum_{n=3}^{\infty}\frac{1}{n(n+3)}=\frac{1}{3}\sum_{n=3}^{\infty}\left(\frac{1}{n}-\frac{1}{n+3}\right)=$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+5}\right)\Rightarrow S_n=\frac{1}{6}-\frac{1}{3n+15}$$

$$\lim_{n\to\infty}S_n=\frac{1}{6}.$$

Exercise :

Find a value of N such that S_N approximates the series with an error at most 10^{-5} where

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)(n+3)}$$

$$\frac{1}{(n+1)(n+3)(n+4)} < \frac{1}{10^5} \Rightarrow 10^5 < (n+1)(n+3)(n+5)$$

$$|S_N - S| < a_{n+1}$$

$\Rightarrow N = 44$ works

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Just check with numbers
