Worksheet 15:

Exercise 1:

(a) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ → Maclaurin

(b) $f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!} (x-a)^n$ → Taylor.

(c). $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$.

$f'(x) = 2 + 3 \cdot 2x + 4 \cdot 3 \cdot x^2 + 5 \cdot 4x^3$

$f''(x) = 3 \cdot 2 + 4 \cdot 3 \cdot 2 \cdot x + 5 \cdot 4 \cdot 3 \cdot x^2$

$f^{(3)}(x) = 4! + 5!\,x.$

$f^{(4)}(x) = 5!$, $f^{(j)}(x) = 0$ for $j > 4$.

Then $f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} \cdot x^4$

$= 1 + 2x + \frac{3!}{2!} x^2 + \frac{4!}{3!} x^3 + \frac{5!}{4!} x^4 = 1 + 2x + 3x^2 + 4x^3 + 5x^4$

(d) $f(x) = 1 + 2x + 3x^2 + 4x^3$ centered at $x = 1$.

$f'(x) = 2 + 3!\,x + 4 \cdot 3x^2$

$f''(x) = 3! + 4!\,x$

$f^{(3)}(x) = 4!$

$f^{(j)}(x) = 0$ for $j > 3$

$\Rightarrow$ $f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$

$= 10 + 20(x-1) + 15(x-1)^2 + 4(x-1)^3$

$= 1 + 2x + 3x^2 + 4x^3$.

Exercise 2:

(a) $f(x) = \ln(1+x)$

$f^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!$

$f'(x) = (1+x)^{-1}$

$f''(x) = -1(1+x)^{-2}$

$f^{(3)}(x) = (-1)^2 \cdot 2!\,(1+x)^{-3}$

$f^{(4)}(x) = (-1)^3 \cdot 3!\,(1+x)^{-4}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$

$\Rightarrow \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!\,x^n}{n!} + f(0)$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$

Ratio test: Converges on $(-1, 1)$.

(b) $f(x) = xe^{2x}$

$f'(x) = 2e^{2x}$

$f^{(n)}(0) = 2^n$

$f''(x) = 2^2 e^{2x}$

$f^{(n)}(x) = 2^n e^{2x}$

$xe^{2x} = xf(x) = x\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+1} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{(n-1)!} x^n$

Ratio test: $\lim_{n\to\infty} \left| \frac{2^n x^{n+1}}{n!} \cdot \frac{(n-1)!}{2^{n-1} x^n} \right| = \lim_{n\to\infty} \left| \frac{2x}{n} \right| = 0 < 1$

The series converges on $(-\infty, \infty)$

Exercise 3:

Use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!}$

a) $f(x) = \frac{x^2}{1-3x} =$

$= x^2 \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^{n+2}$

$= \sum_{n=2}^{\infty} 3^{n-2} x^n.$

converges for $|x| < \frac{1}{3}$.

(b) $e^x + e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} x^n$

(c) $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$

(d) $f(x) = x^5 \sin(3x^2) =$

$= x^5 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (3x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1}}{(2n+1)!} \cdot x^{4n+7}.$

(e) $f(x) = \sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2x)^{2n}}{(2n)!} =$

$= \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n} \cdot x^{2n}}{(2n)!} = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot 2^{2n} \cdot x^{2n}}{(2n)! \cdot 2}$

$$= \frac{1}{2} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^{2n} \cdot x^{2n}}{(2n)! \cdot 2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^{2n-1}}{(2n)!} \cdot x^{2n}$$

Exercise 4:

(a) $f(x) = e^{5x}$, $a = 0$

$f^{(n)}(0) = 5^n$.

$f'(x) = 5e^{5x}$

$f''(x) = 5^2 e^{5x}$

$f^{(n)}(x) = 5^n e^{5x}$

$$e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{5^n}{n!} \cdot x^n.$$

(b). $f(x) = \sin(\pi x)$. $a=1$

$\sin(\pi) = 0$

$\cos(\pi) = 1$

$f'(x) = \pi \cdot \cos(\pi x)$.

$f''(x) = -\pi^2 \cdot \sin(\pi x)$.

$f^{(III)}(x) = -\pi^3 \cos(\pi x)$

$$\sin(\pi x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\pi^{2n}}{(2n)!} (x-1)^{2n}$$

$f^{(4)}(x) = \pi^4 \sin(\pi x)$

Exercise 5:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \underbrace{(\sin(x))'}_{\cos(x)} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n+1) \cdot x^{2n}}{(2n+1)\cdot(2n)!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!}$$

Exercise 6:

$$\lim_{x \to 0} \frac{x - \tan^{-1}(x)}{x^3}$$

$$\tan^{-1}(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!} x^2 + \dots$$

$$f(x) = \tan^{-1}(x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$$

$$f''(x) = -(1+x^2)^{-2} \cdot 2x \qquad f''(0) = 0$$

$$f'''(x) = -2\,(1+x^2)^{-2} + 2x \cdot (-1) \cdot (-2) \cdot (1+x^2)^{-3} \cdot 2x$$

$$f'''(0) = -2.$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$ Interval $(-1,1)$

$$\frac{1}{1+x} = d\,\frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

$$\frac{1}{1+x^2} = 1 - (x^2) + (x^2)^2 - (x^2)^3 + (x^2)^4 + \cdots =$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Integrate

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots\cdots$$

$$\lim_{x\to 0} \frac{x - \tan^{-1}(x)}{x^3} = \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \cdots}{x^3} = \lim_{x\to 0} \frac{\cancel{x^3}\left(\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} \cdots\right)}{\cancel{x^3}}$$

$$= \boxed{\frac{1}{3}}$$

Exercise 7:

$$\int_0^1 x\cos(x^3)\,dx =$$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x^3)^{2n}}{(2n)!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{6n}}{(2n)!}$$

Since we need a 6th order polynomial for $\cos(x)$ we get.

$$\cos(x^3) = (-1)^0 \cdot \frac{x^{6\cdot 0}}{(2\cdot 0)!} + (-1)^1 \cdot \frac{x^{6\cdot 1}}{(2\cdot 1)!} + (-1)^2 \cdot \frac{x^{6\cdot 2}}{4!} + (-1)^3 \cdot \frac{x^{18}}{6!}$$

$$+ (-1)^4 \cdot \frac{x^{24}}{8!} + (-1)^5 \frac{x^{30}}{10!} + (-1)^6 \cdot \frac{x^{36}}{12!}$$

$$\cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \frac{x^{24}}{8!} - \frac{x^{30}}{10!} + \frac{x^{36}}{12!}$$

$$\int_0^1 \left(x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \frac{x^{25}}{8!} - \frac{x^{31}}{10!} + \frac{x^{37}}{12!} \right) dx =$$

$$= \left(\frac{x^2}{2} - \frac{x^8}{8\cdot 2!} + \frac{x^{14}}{14\cdot 4!} - \frac{x^{20}}{20\cdot 6!} + \frac{x^{26}}{26\cdot 8!} - \frac{x^{32}}{32\cdot 10!} + \frac{x^{38}}{38\cdot 12!} \right)\Bigg|_0^1$$

$$\boxed{= \frac{1}{2} - \frac{1}{8\cdot 2!} + \frac{1}{14\cdot 4!} - \frac{1}{20\cdot 6!} + \frac{1}{26\cdot 8!} - \frac{1}{32\cdot 10!} + \frac{1}{38\cdot 12!}}$$

Exercise 8 ;

$f(x) = e^x \ln(1-x)$. centered at 0.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$g(x) = \ln(1-x)$

$g'(x) = (-1)(1-x)^{-1}$

$g''(x) = (-1)\cdot(-1)\cdot(1-x)^{-1}\cdot(-1) = (-1)(1-x)^{-2}$

$g'''(x) = (-1).(-2)(1-x)^{-3}\cdot(-1) = (-2)(1-x)^{-3}$.

$g(0) = \ln(1-0) = \ln(1) = 0$.

$g'(0) = (-1)\cdot(1-0)^{-1} = -1$

$g''(0) = (-1)\cdot(1-0)^{-1} = -1$

$g'''(0) = (-2)(1-0)^{-3} = -2$.

$\ln(1-x)$ has Taylor Series $-x - \frac{x^2}{2} - \frac{2x^3}{3!} + \cdots =$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$

$e^x \ln(1-x)$ has a Taylor series

$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(-x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots\right) =$

$$= 1 - x - \frac{x^2}{2} - \frac{x^3}{3} - x^2 - \frac{x^3}{2} - \frac{x^3}{2!} + \cdots =$$

$$= -x - \frac{x^2}{2} - x^2 - \frac{x^3}{3} - \frac{x^3}{2} - \frac{x^3}{2} + \cdots =$$

$$= -x - \frac{3x^2}{2} - \frac{x^3}{3} - x^3 + \cdots =$$

$$= -x - \frac{3x^2}{2} - \frac{4x^3}{3} + \cdots$$