Worksheet 14:

Exercise 1: Radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$

a)

Every power series has a radius of convergence R, which is either nonnegative or infinity. Converges absolutely when

$$|x-c| < R.$$

(b) it converges for $|\cos(x)| < 1$ so just avoid multiples of π, that is $x \neq n\pi$ for any integer n.

(since $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1, \dots$)

(c) $c_n = \dfrac{n+1}{n!}$, $\sum_{n=0}^{\infty} c_n x^n$

(d) $1{,}5 + (-1)^{n+1} \cdot (0{,}5) = c_n$.

(e) $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$, $c \neq 0$ $\Rightarrow$ Root test:

$\lim_{n\to\infty} \sqrt[n]{|c_n x^n|} = c \cdot |x| < 1$

$|x| < \frac{1}{c} = R.$ $\Rightarrow$ $\boxed{-\frac{1}{c} < x < \frac{1}{c}}$

7) $f(x) = \frac{5}{1-x}$, $|x|<1$

$$\Rightarrow \sum_{n=0}^{\infty} 5x^n$$

Since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

9) See the lecture.

Exercise 2:

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{4^n}(x-3)^n$

$$\lim_{n\to\infty}\left|\frac{(n+1)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n \cdot (x-3)^n}\right| =$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{4n}\right| \cdot |(x-3)| = \frac{1}{4}(x-3)$$

$$-1 < \frac{1}{4}(x-3) < 1 \Leftrightarrow -4 < x-3 < 4$$

$$\boxed{-1 < x < 7}$$

So the radius of convergence is 4.

Interval of convergence: we know $(-1, 7)$ converges, but what about the endpoints: -1 and 7.

$x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(-1-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n = \sum_{n=0}^{\infty}(-1)^{2n} \cdot n$ which diverges.

$x=7$: $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{4^n} \cdot (7-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n}{4^n} \cdot 4^n = \sum_{n=0}^{\infty} (-1)^n \cdot n$ which diverges

So the interval of convergence is $(-1, 7)$.

(b) $4 \sum_{n=1}^{\infty} \frac{2^n}{n} (4x-8)^n = \sum_{n=1}^{\infty} \frac{2^{n+2}}{n} (4x-8)^n$.

$$\lim_{n\to\infty} \left| \frac{2^{n+3} \cdot (4x-8)^{n+1}}{n+1} \cdot \frac{n}{2^{n+2} \cdot (4x-8)^n} \right| = \lim_{n\to\infty} \left| \frac{2n}{n+1} \right| \cdot |4x-8|$$

$$= 2 \cdot |4x-8|$$

So $-1 < 8x - 16 < 1$

$15 < 8x < 17$

$\boxed{\frac{15}{8} < x < \frac{17}{8}}$ $\Rightarrow$ So the radius of convergence is $1/8$.

Interval of convergence: We know $\left(\frac{15}{8}, \frac{17}{8}\right)$ converges, but what about the endpoints.

$x = \frac{15}{8}$: $\sum_{n=0}^{\infty} \frac{2^{n+2}}{n} \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{4(-1)^n}{n}$ which converges by AST.

$x = \frac{17}{8}$: $\sum_{n=0}^{\infty} \frac{2^{n+2}}{n} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{4}{n}$ which diverges (Remember $\sum_{n=0}^{\infty} \frac{1}{n}$)

So the interval of convergence is $\left[\frac{15}{8}, \frac{17}{8}\right)$.

(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$, $\lim_{n\to\infty} \left| \frac{(x^2)^{n+1}}{(-3)^{n+1}} \cdot \frac{-3^n}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{-3} \right|$ we want

$\frac{x^2}{3} < |1| \Rightarrow x^2 < 3 \Rightarrow -\sqrt{3} < x < \sqrt{3}$.

Radius of convergence is $\sqrt{3}$. Now we need to check $\pm\sqrt{3}$ for convergence.

$\underline{x=\sqrt{3}}$: $\sum_{n=0}^{\infty} \frac{((\sqrt{3})^2)^n}{(-3)^n} = \sum_{n=0}^{\infty} \left(\frac{3}{-3}\right)^n = \sum_{n=0}^{\infty} (-1)^n$ diverges.

$\underline{x=-\sqrt{3}}$; $\sum_{n=0}^{\infty} \left(\frac{(-\sqrt{3})^2}{-3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{3}{-3}\right)^n = \sum_{n=0}^{\infty} (-1)^n$ diverges.

So the interval of convergence is $(-\sqrt{3}, \sqrt{3})$.

(d) $\sum_{n=0}^{\infty} n!\,(x-2)^n$ $\lim_{n\to\infty} \left| \frac{(n+1)! \cdot (x-2)^{n+1}}{n! \cdot (x-2)^n} \right| = \lim_{n\to\infty} |n+1|\,|x-2| = \infty$

So the series diverges for all x and the radius of convergence is 0 and the interval of convergence is 0.

(e) $\sum_{n=0}^{\infty} (5x)^n$ converges for $|5x| < 1 \Rightarrow |x| < \frac{1}{5}$

$R = \frac{1}{5}$, Interval $(-\frac{1}{5}, \frac{1}{5})$.

(f) $\sum_{n=0}^{\infty} \sqrt{n}\, x^n$, Ratio test; $\lim_{n\to\infty} \left| \frac{\sqrt{n+1} \cdot x^{n+1}}{\sqrt{n} \cdot x^n} \right| = \lim_{n\to\infty} |x| < 1 = R.$

Interval $(-1, 1)$.

(g). $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$, Ratio test $\lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| = \lim_{n\to\infty} |x| < 1 = R.$

Interval $[-1, 1)$.

(h) $\sum_{n=3}^{\infty} \frac{x^n}{3^n \cdot \ln n}$, $\lim_{n\to\infty} \left| \frac{x^{n+1}}{3^{n+1} \cdot \ln(n+1)} \cdot \frac{3^n \ln(n)}{x^n} \right| = \left| \frac{x}{3} \right| < 1$, $R = 3$

Interval $[-3, 3)$.

(i) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$, Root test $\lim_{n\to\infty} \sqrt[n]{\left| \frac{x-2}{n} \right|^n} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0$

$R = \infty$ interval $(-\infty, \infty)$

(j) $\sum_{n=4}^{\infty} \frac{(-1)^n \cdot x^n}{n^4}$, Ratio test, $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^4} \cdot \frac{n^4}{x^n} \right| = \lim_{n\to\infty} |x| = |x| < 1.$

$R = 1$, Interval $[-1, 1]$.

(k) $\sum_{n=3}^{\infty} \frac{(5x)^n}{n^3}$ Ratio test $\lim_{n\to\infty} \left| \frac{(5x)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(5x)^n} \right| = |5x| < 1$

$R = \frac{1}{5}$, Interval $\{ -\frac{1}{5}, \frac{1}{5} \}$.

Exercise 3: $\int_0^x \frac{1}{1+t^2}\, dt = \arctan(x)$. $\frac{1}{1-(-t^2)} = \frac{1}{1+t^2}$.

$f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot (x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$.

$\arctan x = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt = \boxed{\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = \arctan(x)}$

or $\arctan(x) = \int \frac{1}{1+x^2}\, dx = \int \left(\sum_{n=0}^{\infty} (-1) x^{2n} \right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n+1} + C$

let $x = 0 \Rightarrow 0 = 0 + C$ so $C = 0$.

Exercise 4: $\int_0^x \frac{dt}{1+t} = \ln(1+x)$, use $\frac{1}{1+x} = \frac{1}{1-(-x)}$

thus $\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^n \right) dt = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1}$

Exercise 5:

$$(1+x^2)^{-2} = \frac{1}{(1+x^2)^2}$$

We know $\left(\frac{1}{1+x^2}\right)' = \left((1+x^2)^{-1}\right)' = (-1)\cdot 2x \cdot (1+x^2)^{-2} = \frac{-2x}{(1+x^2)^2}$

We know $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$

$$\frac{-2x}{(1+x^2)^2} = \left(\frac{1}{1+x^2}\right)' = \sum_{n=1}^{\infty} (-1)^n \cdot 2n \cdot x^{2n-1}$$

$$\frac{1}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{\cancel{2}n \cdot x^{2n-1}}{-\cancel{2}x} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot n \cdot x^{2n-2}$$