# Worksheet 13:

## Exercise 1:

a) Assume that the following limit exists: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$

if $\rho<1$, then the series $\sum a_n$ converges absolutely.

if $\rho>1$ then $\sum a_n$ diverges.

if $\rho=1$ the test is inconclusive.

b) Assume the following limit exists: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$.

if $L<1$ then the series $\sum a_n$ converges absolutely.

if $L>1$ then $\sum a_n$ diverges.

if $L=1$ the test is inconclusive.

## Exercise 3:

a) $\sum_{n=0}^{\infty}\left(\frac{3n^3+2n}{4n^3+1}\right)^n$, Use Root test,

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{3n^3+2n}{4n^3+1}\right)^n} = \lim_{n\to\infty}\frac{3n^3+2n}{4n^3+1} = \frac{3}{4}<1$$

, so it converges absolutely.

c) $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{n!}$ Use Ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} \cdot (n+1)^2}{(n+1)!} \cdot \frac{n!}{2^n \cdot n^2} \right| = \lim_{n\to\infty} \left| \frac{2^{\cancel{n+1}}}{\cancel{2^n}} \cdot \frac{\cancel{n!}}{(n+1) \cdot \cancel{n!}} \cdot \frac{(n+1)^2}{n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0$$ so it converges absolutely

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d) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$, Use Ratio test.

$$\lim_{n\to\infty} \left| \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{e^{n+1}}}{\cancel{e^n}} \cdot \frac{\cancel{n!}}{(n+1) \cdot \cancel{n!}} \right| = \lim_{n\to\infty} \left| \frac{e}{n+1} \right| = 0$$

So, it converges absolutely.

~~e)~~ $\cancel{\sum_{n=1}^{\infty} \frac{5^n}{(11-\cos^2(n))^n}}$ ~~$a_n$ Use limit test.~~

~~Since $-1 \le \cos^2(n) \le 1$ then~~

~~Pick $b_n = \frac{5^n}{11^n} = \left(\frac{5}{11}\right)^n \Rightarrow \sum_{n=1}^{\infty}\left(\frac{5}{11}\right)^n$ converges absolutely since $\frac{5}{11} < 1$.~~

$$\cancel{\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5^n}{11^n\left(1-\frac{\cos^2(n)}{11}\right)^n} \cdot \frac{11^n}{5^n} = 1 < \infty}$$

~~So it converges because~~

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f) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$

- Alternating

- $\lim_{n\to\infty} \frac{1}{\sqrt[5]{n}} = 0$

By AST it converges.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt[5]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{5}}}$$ , since $p = \frac{1}{5} < 1$ it diverges.

So it converges conditionally.

g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

•) Alternating.

•) $\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$ By AST converges.

We need to check $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\ln(n+1)} \right| = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

Since for $n \geq 2$, $\ln(n+1) \leq n+1$ then

$0 \leq \frac{1}{n+1} \leq \frac{1}{\ln(n+1)}$ $\Rightarrow$ so it diverges $\Rightarrow$

it converges conditionally.

Exercise 2:

a) No, the Divergence Test does not test convergence.

b) Yes, look definition.

c) False, when limit is 1, the test is inconclusive.

d) Yes

e) Yes.

e) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{(11-\cos^2(n))^n}$

Since $-1 \le \cos(n) \le 1$

then $0 \le \cos^2(n) \le 1$

So this means that $11-\cos^2(n) \geq 10$

$$\Rightarrow \frac{1}{11-\cos^2(n)} \le \frac{1}{10} \Rightarrow$$

$$\Rightarrow 0 < \frac{5}{11-\cos^2(n)} \le \frac{5}{10}$$

$$0 < \frac{5^n}{(11-\cos^2(n))^n} \le \frac{5^n}{10^n} = \left(\frac{5}{10}\right)^n$$

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{10}\right)^n$ Geometric series, $\frac{5}{10} < 1$, thus convergent

So $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{(11-\cos^2(n))^n}$ is convergent.