Worksheet 11:

Exercise 1:

a) The test for divergence only tests for divergence. It does not test for convergence.

b) If $\sum a_n$ converges and $0 \leq b_n \leq a_n$ for all n, then $\sum b_n$ converges too. if $\sum a_n$ diverges and $0 \leq a_n \leq b_n$ for all n, then $\sum b_n$ diverges too.

c) By comparison test, $\sum x_n$ converges. No conclusion if $\sum y_n$ diverges.

d) If a_n and b_n are positive sequences and

$$L = \lim_{n \to \infty} \frac{a_n}{b_n}$$ then

- If $0 < L < \infty$ then $\sum a_n$ and $\sum b_n$ converge or diverge together
- If $L = \infty$ and $\sum a_n$ converges then $\sum b_n$ converge or diverge together.
- If $L = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges.
- If $L = 0$ and $\sum b_n$ converges then $\sum a_n$ converges.
- If $L = 0$ and $\sum a_n$ diverges then $\sum b_n$ diverges.

Exercise 2:

a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1}$ Comparison test, $\frac{1}{n^{3/2}+1} \leq \frac{1}{n^{3/2}}$

converges.

b) $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}$ Comparison test

$\frac{2}{\sqrt{n^2+2}} \geq \frac{2}{\sqrt{n^2+3n^2}} = \frac{1}{n}$. diverges.

$\sum \frac{2}{\sqrt{n^2+2}} \geq \sum \frac{2}{\sqrt{n^2+3n^2}} = \sum \frac{1}{n}$.

c) $\sum_{n=1}^{\infty} \frac{2^n}{2+5^n} \leq \sum \frac{2^n}{5^n} = \sum \left(\frac{2}{5}\right)^n$ converges.

Comparison to geometric series.

d) $\sum_{n=0}^{\infty} \frac{4^n+2}{3^n+1} \leq \sum_{n=0}^{\infty} \frac{4^n+4^n}{3^n} = \sum 2 \cdot \frac{4^n}{3^n}$ diverges.

e) $\sum_{n=1}^{\infty} \left(\frac{10}{n}\right)^{10} = 10^{10} \sum_{n=1}^{\infty} \frac{1}{n^{10}}$ Converges.

f) $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}} \leq \sum \frac{n+n}{n^2\sqrt{n}} = \sum 2 \cdot \frac{n}{n^2\sqrt{n}} = 2\sum \frac{1}{n^{3/2}}$ converges.

g) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{3n^2+14n+7}$ test for divergence $\lim_{n\to\infty} a_n = \frac{1}{3}$. $\Rightarrow$ diverges.

h) $\sum_{n=0}^{\infty} \frac{1+2^n}{2+5^n} \leq \sum_{n=0}^{\infty} \frac{2^n+2^n}{5^n} = 2\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$ converges.

Comparison.

i) $\sum_{n=1}^{\infty} \frac{2}{n^2+5n+2} \leq \sum_{n=1}^{\infty} \frac{2}{n^2}$ converges.

j) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ limit compare to $\sum \frac{1}{n}$

$\lim_{n\to\infty} \frac{\frac{e^{1/n}}{n}}{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}} = 0.$ So the series diverges.

k) $\sum_{n=0}^{\infty} \frac{n}{n^2 - \cos^2 n} \geq \sum \frac{j}{j^2} = \sum \frac{1}{j}$ diverges.

l) $\sum_{n=1}^{\infty} \frac{n!}{n^4}$ $\lim_{n\to\infty} \frac{n!}{n^4} = \infty$ $\Rightarrow$ diverges.

m) $\sum_{n=0}^{\infty} \frac{n^2}{(n+1)!}$ $\Rightarrow$ converges.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx \sum_{n=1}^{10} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{10^2} \approx$

$\approx 1,549$

estimate the value of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the 10^{th} partial sum

b) upper bound for error

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to \infty} \int_{10}^{b} \frac{1}{x^2}\, dx = \lim_{b \to \infty} -\frac{1}{x}\Big|_{10}^{B} =$$

$$= \lim_{B \to \infty} -\frac{1}{b} + \frac{1}{10} = \frac{1}{10}.$$

c) How large k need to be for the error to be $< 0,0001$.

error in $S_k = \sum_{n=1}^{k} \frac{1}{n^2}$

$$R_k \leq \int_{k}^{\infty} \frac{1}{x^2}\, dx = \lim_{b \to \infty} -\frac{1}{x}\Big|_{k}^{b}$$

$$= \frac{1}{k} \leq 0,0001.$$