# Worksheet 10:

## Exercise 1:

(a) No, you can take $(-1)^n$, it is $-1, 1, -1, 1, \ldots$

So it is bounded but $\lim_{n\to\infty}(-1)^n = DNE$.

(b) False, if the series converge then $\lim_{n\to\infty} a_n = 0$.

(c) True, the sum of the series does not change if you add or subtract a finite number of elements.

(d). True, since $c \neq 0$ and $\sum_{n=1}^{\infty} c \cdot a_n$ converges, we can write.

$$\sum_{n=1}^{\infty} c\,a_n = c \cdot \sum_{n=1}^{\infty} a_n$$ so $\sum_{n=1}^{\infty} a_n$ also converges.

(e) True, see (c).

(f) True, it is all zero at the tail.

## Exercise 2:

(a) $\frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \sum_{n=3}^{\infty} \frac{1}{n^2}$.

(b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$

Exercise 3 :

$$S = \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) , \quad S_3 = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) =$$

$$= \frac{1}{2} - \frac{1}{5}.$$

$$S_4 = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) =$$

$$= \frac{1}{2} - \frac{1}{6}.$$

$$S_5 = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) + \left( \frac{1}{6} - \frac{1}{7} \right)$$

$$= \frac{1}{2} - \frac{1}{7}.$$

$$S_n = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{1}{2} - \frac{1}{n+2} \xrightarrow[n \to \infty]{} \frac{1}{2}$$

So $S = \lim_{n \to \infty} S_n = \frac{1}{2}$

Exercise 4:

(a) $\frac{1}{1} + \frac{1}{8} + \frac{1}{8^2} + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^2$

$$S_n = \sum_{n=0}^{n} \left(\frac{1}{8}\right)^n = \frac{1-\left(\frac{1}{8}\right)^n}{1-\frac{1}{8}} = \frac{1-\left(\frac{1}{8}\right)^n}{\frac{7}{8}} = \frac{8}{7}\cdot\left(1-\left(\frac{1}{8}\right)^n\right)$$

$\left(\frac{1}{8}\right)^n \xrightarrow{n\to\infty} 0$

So $\lim_{n\to\infty} S_n = \frac{8}{7}\cdot 1 = \frac{8}{7}$.

(b) $\sum_{n=0}^{\infty}\left(\frac{\pi}{e}\right)^n$, since $\pi = 3{,}14\ldots$, $e = 2{,}7\ldots$ then $\frac{\pi}{e} > 1$

By Geometric series, it diverges.

Exercise 5:

(a) $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ $\quad f(x) = \frac{1}{1+x^2} \Rightarrow$ continuous for $x \geq 0$.

positive since $(1+x^2) \geq 0$ so $\frac{1}{1+x^2} > 0$.

Decreasing. $f'(x) = \frac{-2x}{(1+x^2)^2} \leq 0$ for all $x \geq 0$

So it is decreasing.

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{A\to\infty} \int_0^{A} \frac{1}{1+x^2}\,dx.$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) \quad \text{So} \quad \int_0^{A} \frac{1}{1+x^2}\,dx = \operatorname{arctg}(A) - \operatorname{arctg}(0)$$

$$= \operatorname{arctg}(A) - 0 = \operatorname{arctg}(A).$$

Thus $$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{A\to\infty} \operatorname{arctg}(A) = \frac{\pi}{2}.$$

So By Integral test $A$ is convergent.

(b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$, $f(x) = \frac{x^2}{e^{x^3}}$ is continuous for $n \geq 1$

it is positive and,

$$f'(x) = \frac{2x e^{x^3} - x^2 \cdot x^2 e^{x^3}}{\left(e^{x^3}\right)^2} < 0 \text{ for } n \geq 1$$

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \int_1^{\infty} \frac{x^2}{e^{x^3}}\,dx = \lim_{A\to\infty} \int_1^{A} \frac{x^2\,dx}{e^{x^3}} = \lim_{A\to\infty} \frac{1}{3} \int_1^{A} \frac{d(x^3)}{e^{x^3}}$$

$$= \lim_{A\to\infty} \frac{1}{3} \cdot \left(-e^{-x^3}\right)\Big|_1^{A} = \lim_{A\to\infty} \frac{1}{3}\left(e^{-1} - e^{-A^3}\right) = \frac{1}{3}e^{-1}$$ So

By integration test $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges.

Exercise 6:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ , $f(x) = \frac{1}{x^p}$ positive for $x \geq 1$

it is also continuous since $x \neq 0$.

$f'(x) = \frac{-p}{x^{p+1}} < 0$ for $x \geq 1$.

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{A \to \infty} \int_1^A \frac{dx}{x^p} = \lim_{A \to \infty} \int_1^A x^{-p}\, dx =$$

$$= \lim_{A \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^A = \lim_{A \to \infty} \left( \frac{A^{1-p}}{1-p} - \frac{1}{1-p} \right)$$

if $A < 1$ then $\lim_{A \to \infty} \left( \frac{A^{1-p}}{1-p} - \frac{1}{1-p} \right) = \lim_{A \to \infty} \left( \frac{1}{(1-p) \cdot A^{p-1}} - \frac{1}{1-p} \right)$

$= 0 - \frac{1}{1-p} = \frac{1}{p-1}$ $\Rightarrow$ convergent

if $A > 1$ then $\lim_{A \to \infty} A^{1-p} = \infty$ so divergent.